GEOMETRY OF SECOND ADJOINTNESS FOR p-ADIC GROUPS

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ABSTRACT. We present a geometric proof of second adjointness for a reductive p-adic group. Our approach is based on geometry of the wonderful compactification and related varieties. Considering asymptotic behavior of a function on the group in a neighborhood of a boundary stratum of the compactification, we get a "specialization" map between spaces of functions on various varieties with $G \times G$ action. These maps can be viewed as maps of bimodules for the Hecke algebra, and the corresponding natural transformations of functors lead to the second adjointness. We also get a formula for the "specialization" map expressing it as a composition of the orishperic transform and inverse intertwining operator; a parallel result for D-modules was obtained in [3]. As a byproduct we obtain a formula for the Plancherel functional restricted to a certain commutative subalgebra in the Hecke algebra, generalizing a result by Opdam.

To the memory of Izrail' Moiseevich Gel'fand

CONTENTS

1. Introduction

Parabolic induction and restriction (Jacquet) functors are the basic tool in representation theory of reductive p-adic groups. It follows directly from the definitions that the parabolic induction functor is right adjoint to the Jacquet functor; we will call this ordinary or Frobenius adjointness. It has been discovered by Casselman for admissible representations and generalized by Bernstein to arbitrary smooth representations, that there is another non-obvious adjointness between the two functors. Namely, the parabolic induction functor turns out to be also *left* adjoint to Jacquet functor with respect to the opposite parabolic (we will refer to this as the second or Bernstein adjointness). This fact appears in unpublished notes of Bernstein [1], we reprove it below. Instead of following Bernstein strategy, our approach emphasizes the relation to geometry of the group and related spaces. More precisely, showing that two functors α and β are adjoint amounts to providing morphisms between the identity functor and the compositions $\alpha \circ \beta$, $\beta \circ \alpha$ satisfying certain compatibilities. When α, β are Jacquet and parabolic induction functors, the arrow between the endo-functors of the category of representations of the Levi is easy to define, both for the ordinary and for the second adjointness. To describe the morphisms between the endo-functors of representations of G, recall that an endo-functor of a category of modules usually comes from a bimodule. This is so in our case, moreover, the bimodules can be realized as spaces of functions on a variety equipped with a $G \times G$-action. Thus showing adjointness amounts to providing a map between the function spaces satisfying certain properties. For

the ordinary induction, this turns out to be the map from functions on the group to integral kernels acting on the universal principal series, i.e. on the space of functions on G/U. This well studied map (known as the orispheric transform) is given by an explicit correspondence.

On the contrary, the map responsible for the second adjointness has not, to our knowledge, appeared in the literature. The goal of this paper is a geometric description of this map and its basic properties. It is a map (to be denoted by B) from functions on (the space of F-points of) a certain algebraic variety X to the space of functions on G. Our first observation is that X is an asymptotic cone of G. By this we mean that it admits an open embedding to the normal space $N_{\overline{G}}(Z)$ where $\overline{G}$ is the De Concini-Procesi compactification of G and $Z \subset \overline{G}$ is a $G \times G$ orbit.[1] The map B is uniquely characterized by the property of being *asymptotic to identity* and $G \times G$-equivariant, see Corollary 5.2 (this is also an outgrowth of an idea communicated to us by Bernstein).

The second main result of the paper answers the question of whether this map is given by an explicit correspondence. The answer is that the correspondence giving B can be expressed in terms of the *inverse* of the intertwining operator (Radon transform), see Theorem 7.5. As an application of this result we give a generalization of a result of Opdam which describes the Plancherel functional restricted to a certain commutative subalgebra in the Hecke algebra.

The paper is structured as follows. In section 2 we collect necessary algebro-geometric facts about the wonderful compactification and related varieties. Section 3 develops the formalism of transferring functions between a p-adic variety and the normal bundle to its subvariety. Section 4 contains some algebraic preliminaries. Those sections are mutually logically independent. In section 5 the results of the earlier sections are combined to get the desired map B. In section 6 second adjointness is deduced. In section 7 we formulate some properties of the map B, including the formula expressing it in terms of the inverse to the intertwining operator. In sections 8 and 9 we prove some statements stated in section 7. The last section 10 contains a generalization of Opdam's formula. In the Appendix (joint with Y. Varshavsky) we describe a version of the normal bundle construction which allows to extend some statements about De Concini - Procesi wonderful compactification to reductive groups with a nontrivial center.

We did not attempt to reach a maximal reasonable generality. In particular, Theorem 7.5 should admit a parabolic version which we do not discuss in this paper.

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[1]When the group G is not of adjoint type the compactification is not smooth. In this case we use a modification of the notion of normal cone introduced in the Appendix.

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2. Algebraic varieties related to a semi-simple group

In this section we collect algebro-geometric facts to be used in the main construction. Until the end of the section we work over an arbitrary field which we denote by F. In this section topological notions refer to Zarisky topology.

2.1. **Standard notations.** Let G be a split reductive algebraic group over F. Let $\mathcal{B}$ be the space of Borel subgroups of G. and W be the (abstract) Weyl group. Let $\mathbb{T}$ be the abstract Cartan of G, thus $\mathbb{T}$ is canonically identified with B/U_B for any $B \in \mathcal{B}$, where U_B is the unipotent radical of B. The torus $\mathbb{T}$ acts on the right on G/U_B for any $B \in \mathcal{B}$.

We denote by $X_*(\mathbb{T})$ the lattice of cocharacters of the torus $\mathbb{T}$ and by $X^*(\mathbb{T})$ its lattice of characters. By $X_*(\mathbb{T})^+$, $X^*(\mathbb{T})^+$ we denote the subsets of dominant (co)characters. Let Σ be the set of vertices of the Dynkin diagram of G. We use the bijection between conjugacy classes of parabolic subgroups in G and subsets in Σ sending a parabolic to the set of $i \in \Sigma$ such that the corresponding root subspace is in the radical of P. For $I \subset \Sigma$ we let $P_I = L_I U_I$ denote (an arbitrarily chosen) representative of the corresponding conjugacy class.

For $i \in \Sigma$ we denote by α_i the corresponding simple root, by $\alpha_i^\vee$ the corresponding simple coroot and by ω_i the corresponding fundamental weight. The abstract Weyl group W acts on $\mathbb{T}$, for $i \in \Sigma$ denote by $s_i \in W$ the corresponding simple reflection and let w_0 denote the longest element of W. For a pair B, B' of Borel subgroups we denote by $w(B, B') \in W$ their relative position, with an alternative notation $B \overset{w}{-} B'$.

2.2. **The spaces** X_I, Y. Fix $B_1, B_2 \in \mathcal{B}$ and set $X_{B_1,B_2} = (G/U_{B_1} \times G/U_{B_2})/(B_1 \cap B_2)$, where $B_1 \cap B_2$ acts diagonally on the right. Given another choice $(B_1', B_2') \in \mathcal{B}^2$ in the same conjugacy class, an element $g \in G$ such that $B_i' = gB_ig^{-1}$ for $i = 1, 2$ is defined uniquely up to right multiplication by an element of $B_1 \cap B_2$. Then the map $y \mapsto yg^{-1}$ induces an isomorphism $G/U_{B_i} \widetilde{\longrightarrow} G/U_{B_i'}$. The induced isomorphism $X_{B_1,B_2} \widetilde{\longrightarrow} X_{B_1',B_2'}$ does not depend on the choice of g, thus conjugacy class of the pair (B_1, B_2) defines X_{B_1,B_2} uniquely up to a unique isomorphism. The conjugacy classes are in bijection with W, thus we get a well defined variety $X_w = X_{B_1,B_2}$, where $w = w(B_1, B_2)$. We will only use two extreme cases for which we fix a different notation: $X = X_{w_0}$ and $Y = X_e$ (where e is the unit element). Fixing $B \in \mathcal{B}$ we get $X = (G/U_B)^2/\mathbb{T}$, $Y = (G/U_B)^2/\mathbb{T}$ where the torus acts via the maps $\mathbb{T} \to \mathbb{T}^2$ given by $t \mapsto (t, w_0(t))$ and $t \mapsto (t, t)$ respectively.

We denote by $p_X : X \to \mathcal{B}^2$, $p_Y : Y \to \mathcal{B}^2$ the natural projections.

Notice that the stabilizer in G of the unit coset (U_{B_1}, U_{B_2})-$\mathrm{mod}(B_1 \cap B_2)$ coincides with the stabilizer of its image in $\mathcal{B}^2$. It follows in particular that

$$p_Y^{-1}(\Delta_{\mathcal{B}}) \cong \mathcal{B} \times \mathbb{T}^2/\mathbb{T}_{w_0} \cong \mathcal{B} \times \mathbb{T}, \qquad (2.2.1)$$

$$p_X^{-1}(\mathcal{B}_0^2) \cong \mathcal{B}_0^2 \times \mathbb{T}^2/\mathbb{T}_e \cong \mathcal{B}_0^2 \times \mathbb{T}, \qquad (2.2.2)$$

where $\Delta_{\mathcal{B}} \subset \mathcal{B}^2$ is the diagonal, $\mathcal{B}_0^2 \subset \mathcal{B}^2$ is the open G-orbit, and $\mathbb{T}_{w_0}$, $\mathbb{T}_e \subset \mathbb{T}^2$ is the graph of w_0 and of the diagonal map respectively.

Example 2.1. Let $G = SL(2)$. Then Y parametrizes pairs of nonzero vectors $v_1, v_2 \in F^2$ modulo common dilations, while X is the space of rank one 2×2 matrices. Also, $p_Y^{-1}(\Delta_{\mathcal{B}})$ parametrizes pairs of non-zero colinear vectors, the map $p_Y^{-1}(\Delta_{\mathcal{B}}) \to \mathbb{T} = \mathbb{G}_m$ sends such a pair $(v, \lambda v)$ to λ. The space $p_X^{-1}(\mathcal{B}_0^2)$ parametrizes non-nilpotent rank one matrices, the map $p_X^{-1}(\mathcal{B}_0^2) \to \mathbb{T}$ sends such a matrix to its trace.

2.2.1. *The space X_I.* The definition of X can be generalized as follows. Given two opposite parabolic subgroups P, P^- with unipotent radicals U_P, U_{P^-} one can argue as above that the space $(G/U_P \times G/U_{P^-})/(P \cap P^-)$ depends only on the conjugacy class of P. Thus we get a variety defined uniquely up to a unique isomorphism for every conjugacy class of parabolic subgroups. The latter are in a bijection with subsets $I \subset \Sigma$ (see 2.1), for such a subset I we let $X_I \cong (G/U_{P_I} \times G/U_{P_I^-})/(P_I \cap P_I^-)$ denote the corresponding variety. Thus $X = X_\Sigma$.

Similarly, set $Y_I = (G/U_{P_I} \times G/U_{P_I})/L_I$.

2.2.2. *Radon correspondence.*

Lemma 2.2. *For $w \in W$ there exists a unique orbit $\mathfrak{C}_w$ of the diagonal G^2 action on $X \times Y$, such that*

i) If $(x_1, x_2, y_1, y_2) \in \mathcal{B}^4$ lies in the image of $\mathfrak{C}_w$ under $p_X \times p_Y$ then $x_1 \overset{w}{-} y_1$, $x_2 \overset{w^{-1}w_0}{-} y_2$.

ii) For some (equivalently, for any) $x \in p_X^{-1}(\Delta_{\mathcal{B}})$, $y \in p_Y^{-1}(\mathcal{B}_0^2)$ such that x, y go to the unit element in $\mathbb{T}$ under the projection to the second factor in the decomposition (2.2.1), respectively (2.2.2), we have $(x, y) \in \mathfrak{C}_w$.

We call $\mathfrak{C}_w$ the *Radon correspondence.*

2.2.3. *Radon correspondence is closed.*

Lemma 2.3. *The image of $\mathfrak{C}_w$ under the open embedding $X \times Y \to \bar{X}_{\mathrm{aff}} \times Y$ is closed.*

Proof. Since the subset $\mathfrak{C}_w \subset \bar{X}_{\mathrm{aff}} \times Y$ is $G \times G$-invariant and the group $G \times G$ acts transitively on Y, it is sufficient to show that fibers $\mathfrak{C}_w(y)$ of the projection $\mathfrak{C}_w \to Y$ are closed in $\bar{X}_{\mathrm{aff}}$. The fiber over $y \in Y$ is an orbit of the stabilizer $Stab_{G \times G}(y)$. Without loss of generality we can assume $y \in Y$ is the image of $e \in G \times G$ and therefore $Stab_{G \times G}(y) = T_\Delta \cdot (N \times N) \subset B \times B$.

It is easy to see from the definition of $\mathfrak{C}_w$ that the diagonal Cartan subgroup T_Δ stabilizes the point $\tilde{y} \in \mathfrak{C}_w$ where $\tilde{y}$ is the image of e. Thus this fiber is an orbit of the unipotent group $U \times U$. So the Lemma follows from the general theorem saying that an orbit of the action of a unipotent algebraic group on an affine variety is closed. $\square$

2.3. **Partial compactifications of Y.** The variety Y is a principal bundle over $\mathcal{B}^2$ with structure group $(\mathbb{T} \times \mathbb{T})/\mathbb{T}_\Delta \cong \mathbb{T}$ where $\mathbb{T}_\Delta \subset \mathbb{T} \times \mathbb{T}$ is the diagonal subgroup and the last isomorphism comes from embedding of the first factor to $\mathbb{T} \times \mathbb{T}$.

Recall that $X^*(\mathbb{T})^+$ is the subset of dominant weights in the weight lattice $X^*(\mathbb{T})$. For $w \in W$ set $X^*(\mathbb{T})_w^+ = w(X^*(\mathbb{T})^+)$. We define a partial compactification $\overline{\mathbb{T}}_w$ of $\mathbb{T}$ by: $\overline{\mathbb{T}}_w = Spec(\mathbb{C}[X^*(\mathbb{T})_w^+]) \supset Spec(\mathbb{C}[X^*(\mathbb{T})]) = \mathbb{T}$. We set $\overline{Y}_w = Y \times^{\mathbb{T}} \overline{\mathbb{T}}_w = (Y \times \overline{\mathbb{T}}_w)/\mathbb{T}$, where $\mathbb{T}$ acts diagonally; this is a partial compactification of Y.

2.3.1. *Properness of the closure.* The variety X is well known to be quasi-affine, let $\bar{X}_{\mathrm{aff}}$ be its affine closure.

Let $\bar{\mathfrak{C}}_w$ be the closure of $\mathfrak{C}_w \subset X \times Y$ in $\bar{X}_{\mathrm{aff}} \times \overline{Y}_w$ and $\bar{\mathfrak{C}}'_w$ be the closure of the image of $\mathfrak{C}_w$ in $\bar{X}_{\mathrm{aff}} \times \mathcal{B}^2$ under the projection

$$Id_{\bar{X}_{\mathrm{aff}}} \times p_{1,2}^Y : \bar{X}_{\mathrm{aff}} \times Y \to \bar{X}_{\mathrm{aff}} \times \mathcal{B}^2$$

Lemma 2.4. *a) The natural map $\bar{\mathfrak{C}}_w \to \bar{\mathfrak{C}}'_w$ is an isomorphism.*

b) The projection $\bar{\mathfrak{C}}_w \to \bar{X}_{\mathrm{aff}}$ is proper.

Proof. Since $\mathcal{B}^2$ is proper, b) follows from a). We now check a). The space $\overline{Y}_w$ is easily seen to be a closed subscheme in the fiber product of total spaces of line bundles on $\mathcal{B}^2$ of the form $\mathcal{O}(\lambda, -\lambda)$, $\lambda \in \Lambda_w^+$. The subscheme $\mathfrak{C}_w \subset X \times Y$ is the graph of sections of the pull-back of those line bundles to a subvariety $Z \subset X \times \mathcal{B}^2$; here Z is the preimage under to the projection $X \times \mathcal{B}^2 \to \mathcal{B}^4$ of the subvariety $\{(B_1, B_2, B_3, B_4) \mid B_1 \overset{w}{-} B_3, B_2 \overset{w_0 w^{-1}}{-} B_4\}$. Thus the statement would follows if we show that these sections extend to the closure of Z in $\bar{X}_{\mathrm{aff}} \times \mathcal{B}^2$. Since the variety $\bar{X}_{\mathrm{aff}}$ is normal and $\bar{X}_{\mathrm{aff}} \setminus X \subset \bar{X}_{\mathrm{aff}}$ is of codimension > 1, any section of the line bundle on $X \times \mathcal{B}^2$ extends to $\bar{X}_{\mathrm{aff}} \times \mathcal{B}^2$. Thus it suffices to show that the sections of our line bundles extend to the closure of Z in $X \times \mathcal{B}^2$.

Since the sections are $G \times G$ invariant and X is a homogeneous space, the desired statement follows from the next

Lemma 2.5. *Fix $w \in W$. For a weight $\lambda \in X^*(\mathbb{T})$ the following are equivalent*

i) $\lambda \in X^*(\mathbb{T})_w^+$.

ii) The line bundle $\mathcal{O}(\lambda, -\lambda)|_{\overline{\mathcal{B}_{w_0 w}} \times \overline{\mathcal{B}_w}}$ has a nonzero $U \times U$ invariant section.

Proof. Let us remind a description of the divisor of an U-invariant section of $\mathcal{O}(\lambda)$ on $\mathcal{B}_w$ viewed as a rational section on $\overline{\mathcal{B}}_w$, see e.g. [4, Proposition 1.4.5] and references therein. Namely, for $w \in W$ components of codimension 1 in $\overline{\mathcal{B}}_w \setminus \mathcal{B}_w$ are in bijection with reflections s_α, where α is a positive not necessarily simple coroot, such that $\ell(ws_\alpha) = \ell(w) - 1$. The multiplicity of the corresponding component in the divisor of an U-invariant section of $\mathcal{O}(\lambda)|_{\overline{\mathcal{B}}_w^-}$ is then equal to $\langle \lambda, \alpha \rangle$. Thus those α for which $\ell(ws_\alpha) = \ell(w) + 1$ correspond to elements of $w^{-1}(\Phi)$ which are positive roots; while those α for which $\ell(w_0 w s_\alpha) = \ell(w_0 w) + 1$ correspond to elements of $w^{-1}(\Phi)$ which are negative roots. We see that a nonzero U invariant section exists both for $\mathcal{O}(\lambda)$ on $\overline{\mathcal{B}}_w$ and for $\mathcal{O}(-\lambda)$ on $\overline{\mathcal{B}}_{w_0 w}$ if and only if λ is positive on $w^{-1}(\Phi)$, i.e. $\lambda \in X^*(\mathbb{T})_w^+$.

2.4. **The wonderful compactification.** We introduce a version of De Concini – Procesi wonderful compactification relevant for our purposes. If G is of adjoint type, we let $\overline{G}$ be the wonderful compactification [7] (see also [8] and [13]). If G is an arbitrary reductive group we let G_{ad} denote the quotient of G by its center, $G' \subset G$ be the commutator subgroup, and consider the homomorphism $G \to (G/G') \times G_{ad}$ which is surjective and has a finite kernel. We let $\overline{G}$ denote the normalization of $(G/G') \times \overline{G}_{ad}$ in G.

The components of ∂G are indexed by Σ, for $i \in \Sigma$ let $\overline{G}_i$ be the corresponding component. If G is a product of a torus and an adjoint group, then $\overline{G}$ is smooth and $\partial G = \overline{G} \setminus G$ is a divisor with normal crossings; this is not necessarily true in general.

For $I \subset \Sigma$ set $\overline{G}_I = \bigcap_{i \in I} \overline{G}_i$. It is known that the action of $G \times G$ on $\overline{G}_I$ is transitive and for a parabolic $P_I = L_I U_I$ in the conjugacy class corresponding to I and an opposite parabolic $P^- = L_I U_I^-$ there exists a point in $\overline{G}_I$ with stabilizer $H_I := \{(ul, u^- l^-)|\ u \in U_I,\ u^- \in U_I^-,\ l, l^- \in L_I, l^{-1}l^- \in Z(L_I)^0\}$, where $Z(L_I)^0$ is the identity component in the center $Z(L_I)$ of L_I.

2.4.1. *Normal bundles to strata.* The isomorphism $\overline{G}_I = (G \times G)/H_I$ shows that we have a canonical map $X_I \to \overline{G}_I$ which is a principal bundle with the structure group $Z(L_I)^0$.

For a locally closed smooth subvariety Z in a smooth variety M let $N_M(Z)$ denote the normal bundle. More generally, in the situation described in the Appendix, section 11.1, we let $N_M(Z)$ denote the quasi-normal cone in the sense of 11.1.

If G is adjoint, the normal bundle $N_{\overline{G}}(\overline{G}_I)$ splits canonically as a sum of line bundles $N_{\overline{G}}(\overline{G}_i)|_{\overline{G}_I}$, $i \in I$. Define an action of $Z(L_I)$ on $N_{\overline{G}}(\overline{G}_I)$ such that the action on $N_{\overline{G}}(\overline{G}_i)|_{\overline{G}_I}$, $i \in I$ is by the character α_i.

Claim 2.6. *a) For any G there exists a unique action of $Z(L_I)$ on $N_{\overline{G}}(\overline{G}_I)$ compatible with the above action under the map induced by the isogeny $\overline{G} \to (G/G') \times \overline{G}_{ad}$.*

b) There exists a canonical $G \times G \times Z(L_I)$ equivariant open embedding $X_I \to N_{\overline{G}}(\overline{G}_I)$. Its image is the complement to the divisor $\bigcup\limits_{i \in I} N_{\overline{G}_i}(\overline{G}_I)$.

Remark 2.1. The following description of X_I, though not used explicitly in this paper, is closely related to Claim 2.6. The space X_I is quasi-affine, it is the dense $G \times G$ orbit in its affine closure $\bar{X}_{\text{aff}}$. Thus it suffices to describe the algebra of global regular functions $\mathcal{O}_{gl}(X_I)$. Assume for simplicity that F is of characteristic zero. Then the space $\mathcal{O}_{gl}(G)$ is isomorphic as a $G \times G$-module to: $\bigoplus\limits_{\lambda \in X^*(\mathbb{T})^+} E_\lambda$, where $E_\lambda = V_\lambda \otimes V_\lambda^*$ for the representation V_λ with highest weight λ. Let $m_{\lambda,\mu}^\nu : E_\lambda \otimes E_\mu \to E_\nu$ be the corresponding component of multiplication in $\mathcal{O}_{gl}(G)$. Then $\mathcal{O}_{gl}(X_I) = \mathcal{O}_{gl}(G)$ as $G \times G$ modules and multiplication in $\mathcal{O}_{gl}(X_I)$ is given by: $m_{\lambda,\mu}^\nu(X) = m_{\lambda,\mu}^\nu$ if $\lambda + \mu - \nu$ is trivial on $Z(L_I)$ and $m_{\lambda,\mu}^\nu(X) = 0$ otherwise.

Remark 2.2. The correspondence $\mathfrak{C}_w$ can also be described in terms of geometry of the wonderful compactification $\overline{G}$. Namely, let $\Gamma \subset G \times \mathcal{B}^2$ be the graph of the action of G on $\mathcal{B}$, i.e. it is given by: $\Gamma = \{(g, B_1, B_2)\, g(B_1) = B_2\}$. Let $\bar{\Gamma}$ be the closure of Γ in $\overline{G} \times \mathcal{B}^2$. Recall that $Z \cong \mathcal{B}^2 \subset \overline{G}$ is the closed G^2 orbit. According to [5], [6], the subset

$$(Z \times \mathcal{B}^2)_w := \{(B_1, B_2; B_3, B_4) \mid B_1 \overset{w}{-} B_3, B_2 \overset{w_0 w}{-} B_4\}$$

is an open smooth subscheme in $\bar{\Gamma} \cap Z \times \mathcal{B}^2$. We claim that $\mathfrak{C}_w$ is canonically isomorphic to an open part in the quasi-normal cone $N_{\bar{\Gamma}}((Z \times \mathcal{B}^2)_w)$. The projection $\mathfrak{C}_w \to X$ is the restriction of the natural map (the normal differential of the projection) $N_{\bar{\Gamma}}((Z \times \mathcal{B}^2)_w) \to N_{\overline{G}}(Z) \supset X$. The projection $\mathfrak{C}_w \to Y$ can be described as follows. Recall that there exists a canonical action map $a : \Gamma = G \times \mathcal{B} \to Y$, $a : (g, B) \to (\tilde{B}, g(\tilde{B})\text{-mod}T)$, where $\tilde{B}$ is an arbitrary lifting of $B \in \mathcal{B}$ to a point in G/U. Let us view a as a rational map $\bar{\Gamma} \to \overline{Y}_w$. We claim that $(Z \times \mathcal{B}^2)_w$ is contained in the domain of definition of this map, and $a : (Z \times \mathcal{B}^2)_w \to \mathcal{B}^2 \subset \overline{Y}_w$. Thus the normal differential of a gives a map $N_{\bar{\Gamma}}((Z \times \mathcal{B}^2)_w) \to N_{\mathcal{B}^2}(\overline{Y}_w) = \overline{Y}_w$, which restricts to the projection $\mathfrak{C}_w \to Y$.

We neither prove nor use these statements below.

2.4.2. *Quotient by the U_P action.* Let $P = P_I = L_I U_I$ be a parabolic subgroup whose conjugacy class corresponds to $I \subset \Sigma$. The map $G/U_I \to G/(U_I Z_I)$ is a principal Z_I bundle. If G is adjoint set $A_I = (\mathbb{A}^1)^I$ equipped with the action of Z_I given by the simple roots in the radical of P_I. In general let A_I be the normalization of $(G/G') \times (\mathbb{A}^1)^I$ in the torus $Z(L_I)$. The torus $Z(L_I)$ acts on A_I. Clearly this action has a free open orbit. Thus the associated bundle over $G/(U_I Z_I)$ with fiber A_I, which we denote $\overline{G/U_I} = (A_I \times G/U_I)/Z_I$, contains G/U_I as an open G-orbit

Let $\overline{G}_I^0 \subset \overline{G}_I$ be the preimage of the open U_I orbit under the projection $\overline{G}_I \to G/P_I \times G/P_I^- \to G/P_I$.

Consider the closure of the subgroup Z_I in $\overline{G}$. This is a toric variety for the torus Z_I; let $\tilde{Z}_I$ be the open subvariety in this toric variety which is the union of all Z_I orbits whose closure intersect $\overline{G}_I$. Let $\overline{G}^0(I)$ denote the image of $\tilde{Z}_I$ under the action of $G \times U_I$.

Claim 2.7. *The subset $\overline{G}^0(I) \subset \overline{G}$ is an open subvariety, such that*

i) $\overline{G}^0(I)$ is invariant under the left action of G and the right action of U_I, the action of U_I on $\overline{G}^0_P$ is free.

ii) the quotient $\overline{G}^0(I)/U_I$ is isomorphic to $\overline{G/U_I}$.

iii) We have $G \subset \overline{G}^0(I)$, $\overline{G}^0_I \subset \overline{G}^0(I)$ and the induced embeddings of the quotient spaces coincide respectively with the tautological embedding $G/U_I \to \overline{G/U_I}$ and the embedding $G/(U_I Z_I) \to \overline{G/U_I}$ induced by the embedding $\{0\} \to A_I$.

Example 2.8. Let $G = PGL(2)$, thus $\overline{G} = \mathbb{P}(End(V))$ where $V = k^2$ is the two dimensional vector space. Let $P_I = B_I$ be the group of upper triangular matrices. Then $\overline{G}^0(I)$ is the projectivization of the set of matrices with a nonzero second column. The quotient $\overline{G}^0(I)/U_I$ maps to $\mathbb{P}^1$ by the map sending a matrix to the line of its second column, this identifies $\overline{G}^0(I)/U_I$ with the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(2)$.

2.4.3. *Compactified Bruhat cells.* Fix $B \in \mathcal{B}$. For $w \in W$ let G_w denote the corresponding $B \times B$ orbit. Let $\overline{G_w}$ denote the closure of G_w in $\overline{G}$.

Claim 2.9. [5] *a) The components of the intersection of $\overline{G_w}$ with the closed stratum $\mathcal{B}^2 \subset \overline{G}$ are in bijection with pairs w_1, w_2 such that $\ell(w) + \ell(w_0) = \ell(w_1) + \ell(w_2)$.*

b) Given such a pair (w_1, w_2), the smooth part of the corresponding component is the $B \times B$ orbits $\mathcal{B}_{w_1} \times \mathcal{B}_{w_2}$.

c) The open part of the quasi-normal cone to the smooth locus of the corresponding component is identified with X_{w_1,w_2} where X_{w_1,w_2} is the corresponding $B \times B$ orbit in X.

2.5. **More on partial compactifications of X.**

2.5.1. *Torus closure.* Let $\overline{\overline{\mathbb{T}}} := \overline{\overline{\mathbb{T}}}_1$ denote one of the partial compactifications of the abstract Cartan $\mathbb{T}$ introduced in section 2.3.

Claim 2.10. *a) The closure in $\bar{X}_{\mathrm{aff}}$ of any orbit of the abstract Cartan $\mathbb{T}$ acting on X is isomorphic to $\overline{\overline{\mathbb{T}}}$.*

b) For any w_1, w_2 there exists a $\mathbb{T}$ equivariant isomorphism $\mathbb{T} \cong X_{w_1,w_2}/N^2$. The corresponding projection $X_{w_1,w_2} \to \mathbb{T} = X_{w_1,w_2}/N^2$ extends to a regular map $\overline{X_{w_1,w_2}} \to \overline{\overline{\mathbb{T}}}$ where $\overline{X_{w_1,w_2}}$ denotes the closure in $\bar{X}_{\mathrm{aff}}$.

2.5.2. *Embedding into the matrix space and its compactification.* For a vector space $\mathcal{V}$ let $\mathbb{P}(\mathcal{V})$ denote the projectivization of $\mathcal{V}$ and let $\tilde{\mathbb{P}}(\mathcal{V}) = \mathbb{P}(\mathcal{V} \oplus F) = \mathbb{P}(\mathcal{V}) \cup \mathcal{V}$ be its projective compactification, and set $\mathcal{V}^0 = \mathcal{V} \setminus \{0\}$, $\tilde{\mathbb{P}}(\mathcal{V})^0 = \tilde{\mathbb{P}}(\mathcal{V}) \setminus \{0\}$.

Recall the notation $\bar{X} = N_{\overline{G}}(\overline{G}_\Sigma)$, this is a partial compactification of X in view of Lemma 2.6.

Claim 2.11. *a) Let M be a G-module. There exists a canonical map $\rho_M : X \to End(M)$. This map extends to a map $\overline{\rho_M} : \bar{X} \to \tilde{\mathbb{P}}(End(M))^0$.*

b) There exists a finite collection of G-modules M_i, such that the map $\prod \overline{\rho_{M_i}} : \bar{X} \to \prod \tilde{\mathbb{P}}(End(M_i))^0$ is a closed embedding. This embedding sends the zero section of $N_{\overline{G}}(\overline{G}_\Sigma) = \bar{X}$ to $\prod \mathbb{P}(End(M_i))$.

3. Specializaton to normal bundle for functions on a p-adic manifold

From now on F is a non-Archimedian local field with ring of integers $O \subset F$ and uniformizer π. From now on topological notions are in reference to F-topology on the space of F-points of an algebraic variety over F or its subspaces.

Let W be a smooth analytic variety over a non-archimedian local field F, $D \subset W$ an open subset such that the complement $W - D$ is a union $S = \cup_{i \in \Sigma} S_i$ of smooth divisors S_i with normal crossing. For $I \subset \Sigma$ we define $S_I = \cap_{i \in I} S_i$, $S_I^0 := S_I - \cup_{j \in \Sigma - I} S_{I \cup j}$ and denote by $r_I : N_I \to S_I$ the normal bundle to S_I. For any $J \subset I$ we denote by $N_I^J \subset N_I$ the normal bundle to S_I in S_J. Locally in S_I we can identify N_I with the product $\mathbb{A}^I \times S_I$.

The following result is immediate.

Claim 3.1. *a) We have a canonical direct sum decomposition*

$$N_I = \oplus_{i \in I} N_I^{I-\{i\}}$$

b) $r_J^\star(N_I^J)$ is the normal bundle to $r_J^{-1}(S_I)$ in $r_J^{-1}(S_J)$.

c) The complement $N_I - r_I^{-1}(S_I^0)$ is a union of smooth divisors $r_I^{-1}(S^{I \cup j}), j \in \Sigma - I$ with normal crossing.

Definition 3.1. Let $U \subset W$ be an open subset. We say that an analytic map $\tau_I : U \to N_I$ is *admissible* if

$\tau_{I|S_I \cap U} = Id$

$\tau_I(U \cap S_J) \subset N_I^J$ for $J \subset I$ and

$d\tau_{I|S_I \cap U} = Id.$

Any admissible map $\tau_I : U \to N_I$ defines an embedding $\tau_{IV}^\star : \mathbb{C}(V) \to \mathbb{C}(U)$ for any open subset V of $\tau_I(U) \subset N_I$.

We denote by $(a, y) \to ay$ the natural action of the group $\mathbb{G}_m^I$ on the bundle $\underline{p}_I : N_I \to S_I$ on N_I and define $X_I \subset N_I$ as the open subvariety of points with trivial stabilizer; thus $X_I = N_I \setminus \cup_{j \in I} N_I^{\{j\}}$.

Definition 3.2. a) For any $\lambda \in Hom(\mathbb{G}_m \to \mathbb{G}_m^I) = \mathbb{Z}^I$ we write $z_\lambda := \lambda(t^{-1}) \in (F^\star)^I$. We say that $\lambda \geq \mu$ if $\lambda_i \geq \mu_i$ for all $i \in I$.

b) For any $\lambda \in \mathbb{Z}_+$ we define an operator $T_\lambda \in End(\mathbb{C}(N_I))$ by

$$T_\lambda(f)(y) := f(z_\lambda y)$$

Lemma 3.2. *For any $f \in \mathbb{C}(N_I)$ and a pair $\tau_{I,1}, \tau_{I,2} : U \to N_I$ of admissible maps there exists $\lambda_0 \in \mathbb{Z}$ such that $supp(T^\mu(f)) \subset \tau_{I,1}(U) \cap \tau_{I,2}(U)$ and $\tau_{I,1}^\star(T^m(f)) = \tau_{I,2}^\star(T^\mu(f))$ for all $\mu \geq \lambda$.*

Proof. The following obvious result implies the lemma.

Let S be an analytic F-variety, $\tilde{S} := F^n \times S \to F^n \times S$ and $\tau_I : \tilde{S} \to \tilde{S}$ an analytic morphism such that a) $\tau_I(F^I \times S) \subset F^I \times S$ for any subset $I \subset [1, n]$ where $F^I := \{(a_j) \in F^n | a_j = 0\}$ for $j \notin I$. In particular we have $\tau_I(\{0\} \times S) \subset \{0\} \times S$.

b) $\tau_{I\{0\} \times S} = Id$.

c) $d\tau_{I\{0\} \times S} = Id$.

Consider the sequence of morphsims

$$\phi_n \in Aut(\tilde{S}), \ \phi_n(\tilde{s}) := t^{-n} \tau_I(t^n \tilde{s}), t(v, s) := (tv, s)$$

Claim 3.3. *The restriction of ϕ_λ on any open compact $C \subset \tilde{S}$ is convergent to identity in the analytic topology for $\lambda \to \infty$.*

We restate the result of the Lemma by saying that the *asymptotic embedding*

$$j_I(\lambda) : \mathbb{C}(N_I) \to \mathbb{C}(W), \ f \to \tau_I^\star(T^\lambda(f)), \lambda \gg 0$$

is well defined, i.e. that it does not depend on the choice of an admissible map.

Given two subsets $J \subset I \subset \Sigma$ we can apply 3.2 to the case considered in Claim 3.1 to obtain the asymptotic embedding

$$j_I^J(\tilde{\lambda}) : \mathbb{C}(N_I) \to \mathbb{C}(N_J), f \to \tau_I^\star(T^{\tilde{\lambda}}(f)), \tilde{\lambda} \gg 0$$

where $\tilde{\lambda} \in \mathbb{Z}^I / \mathbb{Z}^J$. The following result is immediate.

Lemma 3.4. *For any $f \in \mathbb{C}(N_I)$ we have*

$$j_J(\mu) \circ j_I^J(\tilde{\lambda})(f) = j_I(\tilde{\lambda} + \mu)(f)$$

for $\tilde{\lambda} \in \mathbb{Z}^I / \mathbb{Z}^J, \mu \in \mathbb{Z}^J, \tilde{\lambda}, \mu \gg 0$ where we use the canonical isomorphism $\mathbb{Z}^I = \mathbb{Z}^I / \mathbb{Z}^J \oplus \mathbb{Z}^J$

The following two results follow easily from the definitions.

Suppose that an analytic unimodular F-group H acts freely on W preserving irreducible components of D. Then for any strata S_I^0 we have a natural action of H on N_I and on N_I^0. We denote by $\pi : W \to \bar{W} := W/H$ the natural projection. Then π defines the projection $\pi_I : N_I \to \bar{N}_I$ where $\bar{N}_I$ is the normal bundle to $\bar{S}_I^0 := S_I^0/H$ in $\bar{W}$. We define push-forwards $\pi_\star : \mathbb{C}(W) \to \mathbb{C}(\bar{W}), \pi_{I\star} : \mathbb{C}(N_I) \to \mathbb{C}(\bar{N}_I)$ as integrations along H.

Lemma 3.5. *The asymptotic embeddings*

$$j_n : \mathbb{C}(V) \to \mathbb{C}(U), \bar{j}_\lambda : \mathbb{C}(\bar{V}) \to \mathbb{C}(\bar{U}), \lambda >> 0$$

are compatible with push-forwards $\pi_\star, \pi_{I\star}$.

Lemma 3.6. *Let $Z \subset W$ be a locally closed smooth subvariety which intersects each of S_I transversely. Then the asymptotic embeddings for W and Z are compatible under the restriction map.*

Let $U_I \subset W, I \subset \Sigma$ be open neighborhoods of S_I and

$$\tau_I : U_I \to N_I, \ I \subset \Sigma, \quad \tau_I^J : \tilde{N}_J^I \to r_J^\star(N_J^I), \ J \subset I$$

be a family of admissible maps (cf. Claim 3.1 b) where $\tilde{N}_J^I \subset N_J$ are open neighborhoods of $r_J^{-1}(S_I)$.

Definition 3.3. We say that a family U_I, τ_I, τ_I^J is *admissible* if for any pair $\emptyset \subset J \subset I \subset \Sigma$ there exists an open neighborhood $U' \subset U$ of S_I in W such that the restriction of $\tau_I^J \circ \tau_J$ on U' coincides with τ_I.

The following result is easy to prove by induction on $|\Sigma|$.

Claim 3.7. *The admissible families exist.*

3.1. **The almost homogeneous case.** In this section we suppose that the variety $S = W - D$ is compact, that an analytic unimodular F-group H acts on W preserving irreducible components of S and that the action on S_I^0 is transitive for all $I \subset \Sigma$. Let $K \subset H$ be an open compact subgroup and U_I, τ_I, τ_I^J be an admissible family as in 3.3.

Lemma 3.8. *For any $s \in S_I$ and a compact subset C of H there exists an open neighborhood U of s in W such that $\tau_I(cKu) = cK\tau_I(u)$ for all $c \in C, u \in U$.*

Proof. We can assume without loss of generality that $U = cK$ for some $c \in H$. The the statement follows by applying Lemma 3.2 to $\tau_{I,1} = \tau_I$ and $\tau_{I,2} = \tau_I \circ c$. $\square$

Proposition 3.9. *For any compact subset C of H there exists an open neighborhood V of S in W, such that $V \subset \cup U_I$ and $\tau_I(hv) = h\tau_I(v)$ for all $h \in C, v \in V, I \subset \Sigma$.*

Proof. It is sufficient to proof the statement separately for any $I \subset \Sigma$. The proof is by the induction on $|\Sigma - I|$.

Consider first the case $I = \Sigma$. Then the variety S_I^0 is compact and therefore is a finite union of K-orbits. Therefore there exist a K-invariant open neighborhood U of S_I in N_I and a finite number of points $u_a \in U, a \in A$ such that any $u \in U \cap D$ can be written in the form

$$u = kT_\lambda u_a, a \in A, k \in K, \lambda \in \mathbb{Z}_+^I$$

The Proposition follows now from Lemma 3.2.

Assume now that the result is known for all $J \subset \Sigma$ containing I. For any $j \in \Sigma - I$ we choose an open neighborhood $\tilde{N}_{I\cup j}^I$ of $r_{I\cup j}^{-1}(S_I)$ as in 3.3. Then therefore there exist a K-invariant open neighborhood U of S_I in N_I and a finite number of points $u_a \in U, a \in A$ such that any $u \in U \cap D - \cup_{j \in \Sigma - I} \tilde{N}_{I\cup j}^I$ can be written in the form

$$u = kT_\lambda u_a, a \in A, k \in K, \lambda \in \mathbb{Z}_+^I$$

The proposition follows now from Lemma 3.2 and the definition of the admissible family.

3.2. **The singular case.** We now drop the assumption that the analytic variety W is smooth and the divisor $S \subset W$ is a divisor with normal crossing. Instead we assume that each stratum S_I is *well approximated* in the sense of the Appendix. The definition of an admissible map applicable to this case is given in the Appendix. Other definitions and results of the present section carry over to this case *mutatis mutandis*.

4. Some algebraic Lemmas

Let $\mathcal{H}$ be a finitely generated Noetherian $\mathbb{C}$-algebra, M a finitely generated $\mathcal{H}$-module, T an automorphism of M over $\mathcal{H}$.

Lemma 4.1. *Any T^{-1}-invariant $\mathcal{H}$-submodule M' of M is T-invariant.*

Proof. Consider the increasing sequence of $T^n M' \subset M, n > 0$. Since $\mathcal{H}$ is Noetherian there exists n such that $T^{n+1} M' = T^n M'$. But then $T M' = T^{-n} T^{n+1} M' = T^{-n} T^n M' = M'$.

Corollary 4.2. *Let $M^0 \subset M$ be a T^{-1}-invariant $\mathbb{C}$-subspace such that $M = \cup_{r>0} T^r M^0$. Then $\mathcal{H} M^0 = M$.*

Proof. Consider $M' := \mathcal{H} M^0 = M$. Then M' is a T^{-1}-invariant $\mathcal{H}$-submodule of M and therefore $T M' = M'$. On the other hand $M = \cup_{r>0} T^r M'$.

Definition 4.1. Given an $\mathcal{H}$-module N and a T^{-1}-invariant $\mathbb{C}$-subspace $M^0 \subset M$ such that $M = \cup_{r>0} T^r M^0$ we say that $\mathbb{C}$-linear map $a : M^0 \to N$ is $\mathcal{H}$-*compatible* if for any $m \in M^0, h \in \mathcal{H}$ there exists an integer $n(m, h) > 0$ such that $ha(T^{-n}m) = a(hT^{-n}m)$ for $n > n(m, h)$.

Lemma 4.3. *Any $\mathcal{H}$-compatible $\mathbb{C}$-linear map $a : M^0 \to N$ extends uniquely to an $\mathcal{H}$-morphism $\tilde{a} : M \to N$.*

Proof. The uniqueness follows from the previous corollary. To show the existence we fix a finite set $m_i, 1 \le i \le s$ of generators of M over $\mathcal{H}$ and consider a left $\mathcal{H}$-submodule L of $\mathcal{H}^s$ given by

$$L := \{(h_i), h_i \in \mathcal{H}, 1 \le i \le s, | \sum_i h_i a(m_i) = 0\}$$

Since the algebra $\mathcal{H}$ is Noetherian there exists a finite set $l^j = \{h_i^j\}, 1 \le i \le s, j \in J$ which generates L as a left $\mathcal{H}$-module.

Given $m \in M$ we want to define $\tilde{a}(m) \in N$. Since a is $\mathcal{H}$-compatible we can find $n > 0$ such that $m_i' := T^{-n} m_i \in M^0, T^{-n} h_i^j m \in M^0$ and $a(h_i^j m_i) = h_i^j a(m_i)$ for all $1 \le i \le s, j \in J$.

As follows from the previous Lemma the set $\{m_i'\}, 1 \le i \le s$ is a set of generators for M.

To define $\tilde{a}(m) \in N$ for $m \in M$ we choose $x_i \in \mathcal{H}$ such that $m = \sum_i x_i m_i'$ and we define $\tilde{a}(m) := \sum_i x_i a(m_i')$. To prove the Lemma we have to show that $\tilde{a}(m)$ does not depend on a choice of a presentation $m = \sum_i x_i m_i'$. In other words we have to show that for any elements $y_i \in \mathcal{H}, 1 \le i \le s$ such that $\sum_i y_i m_i' = 0$ we have $\sum_i y_i a(m_i') = 0$.

Since $l^j = \{h_i^j\}, 1 \le i \le s, j \in J$ is a set of generators of L for M it is sufficient to check that $\sum_i h_i^j a(m_i') = 0$ for all $j \in J$. But by the definition of the set m_i' we have $\sum_i h_i^j a(m_i') = \sum_i a(h_i^j m_i') = a(\sum_i h_i^j m_i') = 0$.

5. Construction and properties of the maps B_I

We switch notation: from now on G, B, $\mathcal{B}$, X etc. will denote the groups/spaces of F-points of algebraic groups or varieties considered in section 2; these are equipped with the p-adic topology.

We apply the construction of section 3 to $W = \overline{G}$ being the wonderful compactification, with $D = G$. If G is adjoint this is a particular case of the situation introduced at the beginning of section 3, otherwise, one should apply considerations of subsection 3.2. We fix an admissible system τ_I.

5.1. **Definition of B_I.** Let $K \subset G$ be a congruence subgroup and

$$\mathcal{H} = \mathbb{C}_c^\infty(K \backslash G / K), \mathcal{H}_L = \mathbb{C}_c^\infty(K \cap L \backslash L / K \cap L)$$

be the corresponding Hecke algebras, $M_I^K := \mathbb{C}(K \backslash X_I / K)$. It is clear that $\mathbb{C}(X_I) = i_I \circ r_I^-(\mathbb{C}(G))$ and it follows from the remark 3.11 in [2] that the $\mathcal{H}^{\otimes 2}$-module M_I^K is finitely generated. We fix a finite set $\{h_q\}, q \in Q$ of $\mathcal{H}$ containing elements h_i^j defined in the proof of the 4.3. We denote by $T \in$

$Aut(M_I^K)$ the action induced by $z_I \in Z_I$. Let $C := \cup_{q \in Q} \text{supp}(h_q) \subset G \times G$ and $V \subset N_I$ be an open neighborhood of S_I^0 as in 3.9 and

$$M_I^0 := \{f \in M_I^K | \text{supp}(f) \subset V\}.$$

The following result is immediate.

Lemma 5.1. *The pair $((M_I^0)^K \subset M_I^K, T)$ satisfies the condition of Corollary 4.2.*

Corollary 5.2. *There exists unique $\mathcal{H}^{\otimes 2}$-covariant map $B_I^K : M_I^K \to \mathcal{H}$ such that $B_I(f) = \tau_I^\star(f)$ for all $f \in M^0$.*

Definition 5.1. We denote by $B_I : \mathbb{C}(X_I) \to \mathbb{C}(G)$ the linear operator whose restriction on $\mathbb{C}(K \backslash X_I / K)$ is equal to B_I^K for all congruence subgroups $K \subset G$.

5.2. The induced map on U_P coinvariants.

Fix a parabolic $P = P_I$. Consider the projection $X_I \to G/P_I \times G/P_I^- \xrightarrow{pr_2} G/P_I^-$. Let X_I^0 be the preimage of the open U_P orbit on G/P_I^- under the composed map. Notice that the right action of U_P on X_I^0 is free and $X_I^0 / U_P \cong G/U_P$ canonically. It follows that $\mathbb{C}(X_I^0)_{U_P} = \mathbb{C}(G/U_P)$.

Likewise, let ${}^0X_I$ be the preimage of the open U_P^- orbit under the composition $X_I \to G/P_I \times G/P_I^- \xrightarrow{pr_1} G/P_I$. Then the U_{P-} action on ${}^0X_I$ is free and ${}^0X_I / U_P^- \cong G/U_{P-}$ canonically, hence $\mathbb{C}({}^0X_I)_{U_{P-}} = \mathbb{C}(U_{P-} \backslash G)$.

Proposition 5.3. *a) The composed map*

$$\mathbb{C}(G/U_P) = \mathbb{C}(X_I^0)_{U_P} \to \mathbb{C}(X)_{U_P} \to \mathbb{C}(G)_{U_P} = \mathbb{C}(G/U_P),$$

where the first arrow is induced by the open embedding and the second one by the map B_I, is equal to identity.

b) The composed map

$$\mathbb{C}(G/U_{P-}) = \mathbb{C}({}^0X_I)_{U_{P-}} \to \mathbb{C}(X)_{U_{P-}} \to \mathbb{C}(G)_{U_P} = \mathbb{C}(U_{P-} \backslash G),$$

where the first arrow is induced by the open embedding and the second one by the map B_I, is equal to identity.

Proof. In view of Lemma 3.5 and Claim 2.7 this map restricted to K-biinvariant functions on a neighborhood of of $G/U_I Z_I \subset \overline{G/U_I}$ comes from an admissible map $\overline{G/U_I} \supset V \to U \subset N_{\overline{G/U_I}}(G/U_I Z_I)$. However, it is easy to see that there exists an algebraic isomorphism

$$N_{\overline{G/U_I}}(G/(U_I Z_I)) \cong \overline{G/U_I}$$

such that the composed map

$$X_I^0 \hookrightarrow N_{\overline{G}}(\overline{G}^0(I)) \to N_{\overline{G/U_I}}(G/(U_I Z_I)) \cong \overline{G/U_I}$$

coincides with the projection $X_I^0 \to G/U_I = G/U_I$. (Here the second arrow is the differential of the projection $\overline{G}^0(I) \to \overline{G/U_I}$.)

Thus identity map $\mathbb{C}(G/U) \to \mathbb{C}(G/U)$ comes from an admissible map for $\overline{G/U_I}$. It follows that the map in part (a) restricted to K-biinvariant functions on some neighborhood of the zero section $G/(U_I Z_I)$ equals identity. Since the map is G equivariant, it is equal to identity on all functions. This proves (a), part (b) is similar. $\qquad\square$

5.3. **The map B_I and K_0 cosets.** Fix a standard maximal open compact subgroup K_0.

Recall the canonical bijections $K_0 \backslash G / K_0 = X_*(\mathbb{T})/W = X_*(\mathbb{T})^+, K_0 \backslash X_I / K_0 = X_*(\mathbb{T})_{W_P} = X_*(\mathbb{T})_I^+$, where W_I is the corresponding parabolic Weyl group and $X_*(\mathbb{T})_I^+$ is the set of coweights positive on simple roots in the Levi. For $\nu \in \Lambda_I^+$ let $(X_I)_\nu$ denote the corresponding K_0^2 orbit.

Lemma 5.4. *Fix a congruence subgroup $K \subsetneq K_0$. There exists $N > 0$ such that for any $\lambda \in \Lambda^+$ satisfying $(\lambda, \alpha) > N$ for all roots α in the radical of $U_I \subset P_I$, the map B_I sends $C^{K \times K}(X_I)_\lambda$ to $C^{K \times K}(G)_\lambda$ and induces an isomorphism $B_I : C^{K \times K}(X_I)_\lambda \xrightarrow{\sim} C^{K \times K}(G)_\lambda$.*

Proof. For varying N these sets form a system of fundamental neighborhoods of the closure S_I of the corresponding stratum. For each stratum S_J in the closure choose a neighborhood V_J of S_J in G. Without loss of generality we can assume that for each J and an admissible system of maps $\phi_J : V_J \to X_J$ the map $B_J|_{C(\phi_J(V_J))}$ coincides with ϕ_J^*. We can also assume that $V_J \subset V$ where V is as in Proposition 3.9 for $C = K_0$. Thus the maps ϕ_J^* on K-biinvariant functions are K_0^2 equivariant.

Fix a maximal torus $T \subset G$. We also fix a Borel subgroup B containing T. Without loss of generality we can assume that the parabolic P_I, $I \subset \Sigma$ contains B. Thus we get an embedding $T \to (G/U_I \times G/U_I^-)/L_I = X_I$. Let ι_I be the composition of this embedding with the embedding $X_*(T) \to T$ sending a cocharacter ν to $\nu(\pi)$; in particular, $\iota_\emptyset = \iota_G$ is the embedding $X_*(T) \to T \subset G$.

The choice of Borel subgroup B defines an isomorphism $T \xrightarrow{\sim} \mathbb{T}$. In view of Lemma 2.10a), the resulting embedding $(\mathbb{G}_m)^\Sigma \to G$ extends an embedding $\overline{\mathbb{T}} \to \overline{G}$. Moreover, the intersection of its image with a stratum $\overline{G}_I$ is a single T-orbit. It is easy to see that there is an admissible system for $\overline{\mathbb{T}}$ where each map is a T-equivariant open embedding sending $\iota_I(\nu)$ to $\iota_J(\nu)$ for $I \subset J$ (here we use that the image of ι_I lies in $N_{(\mathbb{A}^1)^\Sigma}((\mathbb{A}^1)^\Sigma \cap \overline{G}_I) \subset N_{\overline{G}}(\overline{G}_I) \supset X_I$). It also follows from the definitions that under the closed embedding this admissible system is compatible with the one for $\overline{G}$. This implies the Lemma. $\qquad\square$

Remark 5.2. In fact, the proof of the Lemma shows the following stronger statement (which is not used elsewhere in the paper, so we describe it in a remark). Assume that K is normalized by K_0. Then $K \times K$ acts on $K \backslash G_\lambda / K$. It is easy to see that for λ with $(\lambda, \alpha) \gg 0$, the stabilizer of the

point $K\iota(\lambda)K$ in K_0^2 contains $K_0^+ \times K_0^-$, where $K_0^+ = K_0 \cap Rad(P_I)$, $K_0^- = K_0 \cap Rad(P_I^-)$. Since any the group K admits a triangular decomposition: $K = K_+ K_L K_-$ where $K_L = L \cap K$, $K_+ = K \cap Rad(P)$, $K_- = K \cap Rad(P^-)$ it is easy to see that this stabilizer coincides with the stabilizer of the corresponding point in $K\backslash(X_P)_\lambda/K$. Therefore there is a natural K_0^2 equivariant bijection $\Psi_P : K\backslash(X_P)_\lambda/K \widetilde{\longrightarrow} K\backslash G_\lambda/K$. The map $B_P|_{C(X_P)_\lambda}$ coincides with the map induced by that bijection.

6. Second adjointness

6.1. **Basic notations.** We denote by dg the Haar measure on G such that the volume of K_0 is equal to one. For any parabolic subgroup $P_I = L_I U_I$ we denote by du the Haar measure on U_I such that the volume of $U_I(\mathcal{O})$ is equal to one. We denote by dl the Haar measure on L_I such that the imbedding $U_I^- \times L_I \times U_I \to G, (u^-, l, u \to u^- l u)$ is measure preserving. These choices define invariant measures dx, dy and $d\bar{g}$ on Y_I, X_I and G/U_I. In particular we can interpret elements $\phi \in \mathbb{C}(Y_I)$ as operators $\hat{\phi} : \mathbb{C}(G/U_I) \to \mathbb{C}(G/U_I)$ where

$$\hat{\phi}(\psi)(\bar{g}) := \int_{\bar{g}' \in G/U_I} \phi(\bar{g}, \bar{g}') \psi(\bar{g}') dg', \psi \in \mathbb{C}(G/U_I)$$

Definition 6.1. We define the *action map*

$$A_I : \mathbb{C}(G) \to \mathbb{C}(Y_I), \quad f \to A(f)(g_1, g_2) := \int_{u \in U_I} f(g_1 u g_2^{-1})$$

Remark 6.2. In terms of the interpretation in $\mathbb{C}(Y_I)$ as operators, the map A is easily seen to correspond to the action of the Hecke algebra on the universal principal series $\mathbb{C}(Y_I)$.

Let $P = L U_P \subset G$ be a parabolic subgroup and $P^- = L U_P^-$ be the opposite parabolic subgroup. Let i_I, r_I be, respectively, the normalized parabolic induction and the normalized Jacquet functors with respect to P_I and i_I^-, r_I^- be those with respect to P_I^-.

6.2. **The adjunction maps.** For a smooth L_I-module N one defines canonical morphisms

$$can : r_I i_I(N) \to N, \; Can : N \to r_I^- i_I(N);$$

elements of the space of induced representations are N-valued functions on G, and *can* comes from restriction of functions to the closed subset $P_I \subset G$, while *Can* comes from push-forward of functions from the open subset $U_{P_I^-} P_I \subset G$. Using these canonical morphisms one can define for any smooth G-module M and a smooth L_I-module N the follows maps:

$$Hom(M, i_I(N)) \to Hom(r_I(M), N), \quad \phi \mapsto can \circ r(\Phi)$$

$$(6.6.1)$$

$$Hom(i_I(N), M) \to Hom(N, r_I(M^-), N), \quad \psi \mapsto r_I^-(\psi) \circ Can.$$
$$(6.6.2)$$

Frobenius adjointness says that the morphism (6.6.1) is an isomorphism. This is a standard fact, one of possible proofs is as follows. It suffices to define $\alpha : Id \to i_I r_I$ so that the compositions $i_I \to i_I \circ r_I \circ i_I \to i_I$, $r_I \to r_I \circ i_I \circ r_I \to r_I$ are equal to identity. Here the arrows $i_I \to i_I \circ r_I \circ i_I$, $r_I \to r_I \circ i_I \circ r_I$ come from α, and the arrows $i_I \circ r_I \circ i_I \to i_I$, $r_I \circ i_I \circ r_I \to r_I$ come from (6.6.1).

The morphism α comes from the map $A_I : \mathbb{C}(G) \to \mathbb{C}(Y_I)$, since it is easy to see that $\mathbb{C}(Y) \otimes_{\mathcal{H}} M = i_I r_I(M)$, and it is obvious that $\mathbb{C}(G) \otimes_{\mathcal{H}} M = M$. The compatibilites are easy to check.

6.3. **Second adjointness.** We apply a similar strategy to show that (6.6.2) is also an isomorphism.

We have isomorphisms $i_I \circ r_I^-(M) \cong \mathbb{C}(X) \otimes_G M$ and $M \cong \mathbb{C}(G) \otimes_G M$. Thus the map B_I yields for every $M \in Sm(G)$ a map

$$B_I(M) : i_I \circ r_I^-(M) \to M.$$

In other words Bernstein's morphism B_I defines a morphism $\beta : i_I \circ r_I^- \to Id$. Consider the compositions:

$$\nu_I : i_I \overset{Id \otimes Can}{\longrightarrow} i_I \circ r_I^- \circ i_I \overset{\beta \otimes Id}{\longrightarrow} i_I$$

and

$$\tau_I : r_I^- \overset{Can \otimes Id}{\longrightarrow} r_I^- \circ i_I \circ r^-{}_I \overset{Id \otimes \beta}{\longrightarrow} r^-{}_I$$

Theorem 6.1. *a) The morphisms ν_I and τ_I are the identity morphisms.*

b) The map (6.6.2) is an isomorphism. In particular, i_I is the left adjoint of r_I^-.

Proof. (b) follows from (a) by a standard argument. To check (a) observe that the maps of functors ν_I, τ_I come from the maps of bimodules considered, respectively, in parts (a) and (b) of Proposition 5.3. Thus part (a) of the Theorem follows from that Proposition. □

7. BERNSTEIN MAP AND INTERTWINING OPERATORS

In this section we formulate some properties of the maps introduced above and state a result which says that the maps A and B are connected via the intertwining operator. The proof occupies the next two sections.

7.1. **Bounded supports.** Recall that $\bar{X}_{\text{aff}}$ denotes (the space of F-points of) the affine closure of the quasi-affine algebraic variety X. Let us say that a subset $C \subset X$ is bounded if the closure of C in $\bar{X}_{\text{aff}}$ is compact.

Let $C_b(X)$ denote the space of locally constant functions on X which are supported on a bounded subset in X.

Proposition 7.1. *For $f \in \mathcal{H}$ the support of $B_I^*(f)$ is bounded.*

7.2. **Intertwining operators.** Fix $w \in W$. Let $\mathfrak{C}_w$ be as in Lemma 2.2; we call it the *Radon correspondence.* We define the intertwining operator (or Radon transform) $I_w : C(X) \to C'(Y)$, $I_w : f \mapsto pr_{2*}^w pr_1^{w^*}(f)$ where $pr_1^w : \mathfrak{C}_w \to X$, $pr_2^w : \mathfrak{C}_w \to Y$ are the projections.

Proposition 7.2. *Let $S \subset X$ be a closed bounded subset. Then the map $pr_2^w : (pr_1^w)^{-1}(S) \to Y$ is proper.*

Corollary 7.3. *The intertwining operator I_w naturally extends to a map $I_w : C_b(X) \to C'(Y)$.*

Let us say that a subset in Y is w-bounded if its closure in $\overline{Y}_w$ is compact.

Let $C_b^w(Y)$ denote the space of locally constant functions on Y whose support is w-bounded.

Proposition 7.4. *a) The map I_w sends $C_b(X)$ to $C_b^w(Y)$.*
b) The map $I_w : C_b(X) \to C_b^w(Y)$ is an isomorphism.

7.3. **Main result.**

Theorem 7.5. *For any $w \in W$ we have $A = I_w B^\star$.*

Remark 7.6. The latter equality resembles the result of [3, Corollary 6.2]. More precisely, in *loc. cit.* one finds an isomorphism of two functors between the categories of D-modules. Following a standard analogy between maps of function spaces and functors on the categories of D-modules given by "the same" correspondence one gets that one of the two functors considered in *loc. cit.* is analogous to the map $I_w^{-1} A$. The other functor in [3] is that of *nearby cycles,* or specialization. The characterization of B via specialization of functions on an F-manifold to normal cone makes it natural to consider $B^\star$ as an analogue of specialization functor between the categories of D-modules. It would be interesting to find a precise mathematical statement underlying these heuristic considerations.

The proof of the Theorem is given in the section 9.

Corollary 7.7. *For any $w \in W$ we have $B = A^\star (I_w^\star)^{-1}$.*

8. SOME PROPERTIES OF RADON CORRESPONDENCE

8.1. **Proof of Propositions 7.2, 7.4.** Proposition 7.2 follows from Lemma 2.3.

Proposition 7.4(a) follows from Lemma 2.4.

Proposition 7.4(b) follows from the following more precise statement.

Along with I_w we will also consider the adjoint operator $I_w' = pr_{1*}^w (pr_2^w)^* : C_b(Y) \to C'(X)$.

Proposition 8.1. *There exists elements σ_1, σ_2 in the group algebra of the torus T with the following properties.*

i) Both σ_1 and σ_2 are products of elements of the form $[\alpha_i] - c_i$ where c_i is a constant and $[\alpha_i] \in T$ is a representative of the coset of T^0 corresponding to a positive coroot.

ii) For $f \in C(X)$ the element $I_w(\sigma_1(f))$ has compact support.

iii) For $f \in C(X)$ the element $I'_w I_w(\sigma_1(f))$ (which is well defined by ii) has compact support, and we have $I'_w I_w(\sigma_1(f)) = \sigma_2(f)$.

Proof. Fix a minimal decomposition for elements w, $w^{-1} w_0$. This defines a presentation of I_w as a composition of $\ell(w)$ simpler correspondences. It suffices to prove a similar statement for each of these similar correspondences. This reduces to an elementary property of Radon transform for functions on the plane. $\square$

The rest of the section is devoted to the proof of Proposition 7.1.

Remark 8.1. The above proof does not apply in the case when Borel subgroup is replaced by a parabolic one. It is possible to give an alternative proof admitting such a generalization.

8.2. **Bernstein center and supports.** Let $\mathcal{Z}$ denote the Bernstein center of G. We define an increasing filtration on $\mathcal{Z}$ as follows. Recall [2] that $Spec(\mathcal{Z})$ is the union of connected components where each component has the form $Spec(\mathcal{Z}_k) = T_k/W_k$ where L_k is a Levi subgroup in G, T_{L_k} is the dual torus to L_k/L'_k and W_k is a finite group acting on T_k (in fact, W_k is a subgroup in $T_{L_k} \rtimes W_k$ where W_k is the Weyl group). Thus for every i, the summand $\mathcal{Z}_k$ is a subalgebra in $\mathbb{C}[X_*(L_k/L'_k)]$. We have embeddings $X_*(A_k) \to X_*(L_k/L'_k)$, $X_*(A_k) \to X_*(\mathbb{T})$ where A_k is the center of L_k; the first embedding has a finite index. Thus $X_*(L_k/L'_k) \subset X_*(\mathbb{T})_{\mathbb{Q}} := X_*(\mathbb{T}) \otimes \mathbb{Q}$ canonically.

For $\mu \in X_*(\mathbb{T})_+$ set

$X_*(\mathbb{T})_{\mathbb{Q}}^{\leq \mu} = \{\lambda \in X_*(\mathbb{T})_{\mathbb{Q}} \mid (w(\lambda), \alpha) \leq (\mu, \alpha) \text{ for all simple coroots } \alpha, \ w \in W\}$,

and define

$$(\mathcal{Z}_k)_{\leq \mu} = \mathcal{Z}_k \cap \mathbb{C}[X_*(\mathbb{T})_{\mathbb{Q}}^{\leq \mu}], \quad \mathcal{Z}_{\leq \mu} = \prod (\mathcal{Z}_k)_{\leq \mu}.$$

This equips $\mathcal{Z}_k$, $\mathcal{Z}$ with filtrations indexed by the partially ordered semi-group of dominant weights. It is easy to see that the Rees ring $\bigoplus_\mu \mathcal{Z}_{k \leq \mu}$ are finitely generated is finitely generated for each k.

Proposition 8.2. *For any congruence subgroup $K \subset G$ there exists λ_0 such that for $\lambda \geq \lambda_0$*

$\mathcal{Z}_{\leq \mu} \cdot \mathcal{H}_{\leq \lambda} \subset \mathcal{H}_{\leq \lambda + \mu}.$

Proof. Since the Rees ring $\bigoplus_\mu \mathcal{Z}_{k \leq \mu}$ is finitely generated for each k and $\mathcal{Z}_k \mathcal{H} = \{0\}$ for almost all k, there exists a weight μ such that the components $\mathcal{Z}_{\leq \mu_i}$ generate the rings $\mathcal{Z}_k$ for all k for which $\mathcal{Z}_k \mathcal{H} \neq \{0\}$. Therefore it suffices to show that for any k and sufficiently large λ we have

$$\mathcal{Z}_{k \leq \mu} \cdot \mathcal{H}_{\leq \lambda} \subset \mathcal{H}_{\leq \lambda + \mu_i}. \tag{8.8.1}$$

We will need the following characterization of elements in the Bernstein center $\mathcal{Z}$.

Claim 8.3. *a) For any element $h \in \mathcal{Z}$ there exists an element $h_L \in \mathcal{Z}(L)$ such that the left action of h on $\mathbb{C}(G/U_P)$ coincides with the action of h_L coming from the right action of L.*

b) Furthermore, if $h \in \mathcal{Z}_{\leq\mu}$, then the support $\mathrm{supp}(h_L)$ satisfies the following condition. Let $L_c \subset L$ be generated by compact subgroups, so that $L/L_c \cong X_*(L/L') \subset X_*(\mathbb{T})_{\mathbb{Q}}$. Then the image of $\mathrm{supp}(h_L)$ in L/L_c is contained in $X_*(L/L')_{\leq\mu}$.

We can identify the double quotient $K_0 \backslash X_I / K_0$ with $X_*(\mathbb{T})/W_I = X_*(\mathbb{T})_I^+$, where W_I is the corresponding parabolic Weyl group and $X_*(\mathbb{T})_I^+$ is the set of coweights positive on simple roots in the Levi subgroup L_I. For $\nu \in X_*(\mathbb{T})_I^+$ we denote by $(X_P)_\nu$ the corresponding K_0^2 orbit.

Lemma 8.4. *Fix $I \subset \Sigma$. For any $\mu \in X_*(\mathbb{T})^+$ there exists a finite set S_μ of linear combinations of the form $\sum r_i \alpha_i^\vee$, $i \in I$ such that $\mathcal{Z}_{\leq\mu} C(X_{P_I})_\nu^{K \times K} \subset \sum C(X_{P_I})_\eta$, $\eta \in \nu + \mathrm{conv}(W(\mu)) + S_\mu$ where conv is the convex hull.*

Proof. We have a fibration $X_{P_I} \to G/P_I^-$ with fibers G/U_P; the set $K_0 \backslash G/(U_P \cdot K_0^L)$ maps isomorphically to $K_0 \backslash X_{P_I}/K_0$. The left action of $\mathcal{H}$ we consider here is fiberwise, thus it suffices to prove a similar statement about the action on $\mathbb{C}(G/U_P)$.

An element $h \in \mathcal{H}(L)$ supported on the preimage of a given $\zeta \in X_*(L/L')$ sends $\mathbb{C}(G/U_P)_\nu$ to the sum of $\mathbb{C}(G/U_P)_\eta$ where $\eta \in \nu + \zeta + S$ for some finite subset $S = S(h)$ of linear combinations of $\alpha_i^\vee$, $i \in I$. Thus Lemma follows from Claim 8.3. $\square$

8.2.1. *End of proof of Proposition 8.2.* Fix $N \in \mathbb{Z}_{>0}$ as in Lemma 5.4. We can and will assume that $N > 2r_i$ for r_i as in Lemma 8.4, $\mu \in M$. We can and will assume also that $\langle \lambda, \alpha \rangle > 2N$ for any simple root α.

Fix $\nu \in X_*(\mathbb{T})$, $\nu \leq \lambda$. Let P_ν be the parabolic such that the simple roots in its Levi are exactly those simple roots α for which $(\alpha, \nu) \leq N$. By Lemma 5.4 we have $f = B_P(f_P)$ for some $f_P \in C(X_P)_\nu$. Thus for $z \in Z_{\leq\mu}$ we have: $z(f) = B_P(zf_P)$. In view of Lemma 8.4, $z(f_P) \in \sum C(X_P)_\eta$ with $\eta \in \nu + \mathrm{conv}(W(\mu)) + S_\mu$ for a fixed finite set S_μ of linear combination $\sum r_i \alpha_i$ where α_i are coroots of the Levi.

We have $\nu + \sum r_i \alpha_i \leq \lambda$, because for every simple coroot α_i of the Levi and the corresponding fundamental weights ω_i we have:

$$(\lambda - \nu, \omega_i) > \frac{1}{2}(\lambda - \nu, \alpha_i) > \frac{1}{2}(2N - N) = \frac{1}{2}N > r_i,$$

where the first inequality follows from the fact that $\lambda - \nu$ is a sum of positive coroots, while the other ones follow from the assumptions on N. The claim follows. $\square$

In the next subsection we will need the following consequence of 8.2.

Lemma 8.5. *Fix an open compact subgroup $K \subset K_0$. There exists a finite subset $S \subset X_*(\mathbb{T})$ such that for any $\mu \in X_*(\mathbb{T})_I^+$, $\phi \in C(X_I)_\mu$ and $\lambda \in X_*(\mathbb{T})^+$ and any $w \in W$ there exist $\psi \in C(X_I)$, $h \in \mathcal{H}$, such that $\phi = h(\psi)$, $supp(h) \subset G_{\leq\lambda}$, $supp(\psi) \subset (X_I)_{\mu-w(\lambda)+S}$*

Proof. The space $C(X)$ is acted upon by the maximal torus $\mathbb{T}$; for our current purposes it suffices to consider the action of the subgroup $X_*(\mathbb{T}) \subset \mathbb{T}$ (where the embedding depends on the choice of a uniformizer $\pi \in F$).

It follows from the description of $\mathcal{Z}$ in [2] that the action of an element $s \in \mathbb{C}[X_*(\mathbb{T})]^W$ on $C(X)^{K \times K}$ coincides with an action of some element in $s' \in \mathcal{Z}$. Moreover, if $s \in \mathbb{C}[X_*(\mathbb{T})]_{\leq\mu}$ for some $\mu \in X_*(\mathbb{T})^+$, then s' can be chosen in $\mathcal{Z}_{\leq\lambda}$.

It is clear that $C(X)_0$ generates $C(X)$ as a module over $X_*(\mathbb{T})$. Choose a finite dimensional space of generators for $\mathbb{C}[X_*(\mathbb{T})]$ over $\mathbb{C}[X_*(\mathbb{T})]^W$, then applying it to $C(X)_0^{K \times K}$ we get a space of generators for $C(X)^{K \times K}$ over $\mathbb{C}[X_*(\mathbb{T})]^W$, let us denote it by V. We can also assume without loss of generality that $\mathbb{C}[X_*(\mathbb{T})]_{\leq\lambda}C(X)_0 \subset \mathbb{C}[X_*(\mathbb{T})]_{\leq\lambda}^W V$ for all $\lambda \in X_*(\mathbb{T})^+$.

Let us now choose λ_0 as in 8.2 and choose S be such that $C(X)_S \supset \mathbb{C}[X_*(\mathbb{T})]_{\leq\lambda_0}^W \cdot V$. We claim that this S satisfies the condition of 8.5. Indeed, for $\lambda \geq \lambda_0$ we have

$$C(X)_{\leq\lambda} \subset \mathbb{C}[X_*(\mathbb{T})]_{\leq\lambda}V = \mathbb{C}[X_*(\mathbb{T})]_{\leq\lambda-\lambda_0}^W \mathbb{C}[X_*(\mathbb{T})]_{\lambda_0}^W V \subset \mathcal{H}_{\leq\lambda}C(X)_S.$$

Here in the last inclusion we used the inclusion $\mathbb{C}[X_*(\mathbb{T})]_{\leq\lambda-\lambda_0}^W \delta_K \subset \mathcal{Z}_{\leq\lambda-\lambda_0}\delta_K \subset \mathcal{H}_{\leq\lambda}$ which follows from 8.2. $\qquad\square$

The last ingredient needed in the proof of 7.1 is the following easy statement.

Claim 8.6. *For $\lambda_0 \in X_*(\mathbb{T})$ the subset $X_{\leq\lambda_0} := \cup_{\lambda\leq\lambda_0}X_{I,\lambda}$ has compact closure in $\bar{X}_{aff}$.*

8.3. **Completion of the proof of Proposition 7.1.** Since the space of bounded distributions is clearly invariant under the $G \times G$ action, it suffices to prove the statement for $f = \delta_K$.

In view of Lemma 8.6 it is enough to show the existence of $\lambda_0 \in X_*(\mathbb{T})_I^+$ such that for $\phi \in C(X)^{K \times K}$ the condition $supp(\phi) \subset X_{I,\mu}$, $\mu \not\leq \lambda_0$ implies that $B(\phi)|_1 = 0$.

Assume $\phi \in C(X)_\mu$ and write $\mu = \lambda_1 - \lambda_2$, $\lambda_1, \lambda_2 \in \Lambda^+$. Apply Corollary 8.5 with $w = w_0$. We get that $\phi = h(\psi)$, for some $h \in \mathcal{H}_{\leq\lambda_2'}$ and $supp(\psi) \subset X_{\lambda_1+S}$; here $\lambda_2' = -w_0(\lambda_2)$ is the dual weight. We can and will assume without loss of generality that the pairing of each coweight in $\lambda_1 + S$ with any fundamental weight is larger than N where N is as in Lemma 5.4. This

Lemma implies then that $\mathrm{supp}(B(\psi)) \subset G_{\lambda_1+S}$. Thus $f * B(\psi)|_1 = 0$ for $f \in \mathcal{H}_{\leq\nu}$ unless $\lambda_1 + \sigma \leq \nu'$ for some $\sigma \in S$, $\nu' = -w_0(\nu)$. If we choose λ_0 so that $\lambda_0 \not\leq -\sigma$ for $\sigma \in S$, we get that $\lambda_2 = \lambda_1 - \mu \not\geq \lambda_1 + \sigma$, so $h * B(\psi)|_1 = 0$ since $h \in \mathcal{H}_{\leq\lambda_2'}$. $\qquad\square$

9. Proof of Theorem 7.5

Recall that $C_b(X)$ denotes the space of locally constant compactly supported functions on X.

Proposition 9.1. $r(C_b(X)) \cong \bigoplus_{W \times W} C_b(T)_{w_1,w_2}$.

The Proposition readily follows from the next Lemma. $\qquad\square$

Lemma 9.2. *Let $\beth \subset X$ be a bounded subset. Fix $w_1, w_2 \in W$, and let X_{w_1,w_2} be the corresponding $B \times B$-orbit. Then the map $\beth \cap X_{w_1,w_2} \to T = X_{w_1,w_2}/N^2$ is proper and its image is bounded.*

For every w_1, $w_2 \in W$, every bounded subset in T is the image of $\beth \cap X_{w_1,w_2}$ in T for some bounded subset $\beth \subset X$.

Proof. Without loss of generality we can assume that $\beth$ is invariant under an open compact subgroup in $G \times G$. Then properness in (a) amounts to compactness of the intersection of $\beth$ with every fiber of the projection $X_{w_1,w_2} \cap \beth \to T$, which follows from the fact that the fibers of projection $X_{w_1,w_2} \to X_{w_1,w_2}/N^2$ are closed in $\bar{X}_{\mathrm{aff}}$, being orbits of a unipotent group acting on an affine algebraic variety.

The fact that the image of the projection is bounded and the last statement follow respectively from parts b) and a) of Claim 2.10. $\qquad\square$

9.1. **The subquotient maps for $B^\star$.** In the next statement we use identifications of T-torsors $N\backslash G_w/N$ and $N\backslash X_{w_1,w_2}/N$ where $w_1w_2w_0 = w \in W$. Such an identification follows from 2.9.

Lemma 9.3. *Suppose that $w = w_1w_2w_0$ and $\ell(w) + \ell(w_0) = \ell(w_1) + \ell(w_2)$. Then the composition $r(G)_{\geq w} \xrightarrow{r(B^\star)} r(C_b(X)) \to r(C_b(X))_{w_1,w_2}$ factors through a map $r(G)_w \to r(C_b(X))_{w_1,w_2}$. This map equals the canonical embedding $\mathbb{C}(T) \to C_b(T)$.*

Proof. The composed map in question factors through a map $r(G)_w \to r(C_b(X))_{w_1,w_2}$ because there are no nonzero T^2 equivariant maps $(r(G)_{<w}) \to r(C_b(X))_{w_1,w_2}$.

Let $pr_{w_1,w_2} : X_{w_1,w_2} \to T = X_{w_1,w_2}/N^2$ be the projection. The statement readily follows from the following formula:

$$pr_{w_1,w_2*}(B^\star(f)|_{X_{w_1,w_2}}) = \overline{f}; \qquad\qquad (9.9.1)$$

here $f \in C(G)$ is such that $f|_{G_{<w}} = 0$ and $\overline{f}$ is the image of f in $r(C(G))_w = C(T)$. Notice that the direct image is well defined since the map

$$\mathrm{supp}(B^\star(f)) \cap X_{w_1,w_2} \to T = X_{w_1,w_2}/N^2$$

is proper in view of Lemma 9.2 and Proposition 7.1.

Since the map pr_{w_1,w_2*} factors through coinvariants with respect to the action of $N \times N$, we have $pr_{w_1,w_2*}(B^\star(f)|_{X_{w_1,w_2}}) = F(\overline{f})$ for some $T \times T$ equivariant map $F : C(T) \to C_b(T)$. Thus (9.9.1) would follow if we show that restrictions of the two sides to a non-empty open subset $C \subset T$ (which may depend on K but not on f) coincide. The construction of the map B in section 5 (see Corollary 5.2) makes it clear that for some neighborhood V_G of $Z = \mathcal{B} \times \mathcal{B}$ in $\overline{G}$, a neighborhood V_X of the zero section in the normal bundle $N_{\overline{G}}(Z)$ and an admissible bijection $\tau : V_X \to V_G$ we have $B(f) = \tau_*(f)$ for any $f \in C(X)^{K \times K}$, $\mathrm{supp}(f) \subset V_X$. In view of Lemma 3.5, we will be done if we show that for some $N \times N$ invariant subset $U \subset X_{w_1,w_2}$ we have $U \subset V_X$. This follows from the next geometric Lemma. $\square$

Lemma 9.4. *For a representation M of G let $\rho_M : \bar{X}_{aff} \to End(M)$ be the canonical map as in Claim 2.11.*

a) Suppose that a subset $\mathfrak{Z} \subset X$ is such that for any M the closure of $\rho_M(\mathfrak{Z})$ does not contain zero. Then for any neighborhood V of the zero section in $N_{\overline{G}}(Z) \supset X$ we have $z_I^N(\mathfrak{Z}) \subset V$ for $N \gg 0$.

b) Every orbit of the group $N \times N \times T_0$, where T_0 is a compact subgroup in the abstract Cartan, satisfies the conditions of part (a).

Proof. (a) follows from Claim 2.11, in view of the following easy observation: given finite dimensional vector spaces $\mathcal{V}_i$ over F, and a subset $\mathfrak{Z}$ in $\prod \mathcal{V}_i^0$ such the image of $\mathfrak{Z}$ under the i-th projection does not contain zero in its closure, the closure of $\mathfrak{Z}$ in $\prod \tilde{\mathbb{P}}(\mathcal{V}_i)$ is compact; hence for any neighborhood V of $\prod \mathbb{P}(\mathcal{V}_i)$, multiplication by π^{-N} sends $\mathfrak{Z}$ to V for some N.

To prove (b) notice that we can assume without loss of generality that the neighborhood V is T^0 invariant. Thus it is enough to show the same statement for an $N \times N$ orbit. The image of such an orbit in $End(M)_i$ is also an $N \times N$ orbit different from $\{0\}$. Since an orbit of a unipotent group on an affine algebraic variety is Zariski closed, we get the statement. $\square$

Remark 9.1. The implication in part (a) of the Lemma is actually an "if and only if" statement, we only proved the direction we need to save space.

The last auxiliary fact needed in the proof of the Theorem is the following standard property of intertwining operators.

Claim 9.5. *A component of $r(X)_{w_1,w_2} \to r(Y)_{w_1',w_2'}$ of I_v equals identity if $w_1' = vw_1$, $w_2' = w_2w_0v^{-1}$ and $\ell(w_1) = \ell(v) + \ell(w_1')$, $\ell(w_2) = \ell(w_2') + \ell(w_0v^{-1})$.*

9.2. **Proof of Theorem 7.5.** Comparing Proposition 7.1 with Corollary 7.3 we see that the composition $I_wB^\star$ is well defined.

Then Lemma 9.3 together with Claim 9.5 show that $r(I_wB^\star)$ induces a quotient map $r(C(G))_1 \to r(C_b^w(Y))_{1,1}$ which is equal to the canonical embedding $C(T) \to C_b^w(T)$. By Frobenius adjointness this implies that $I_wB^\star$ is the composition of A with the embedding $C(Y) \to C_b^w(Y)$.

10. Plancherel functional on the abelian subalgebra

In this last section we present an application of the equality $B = A^\star(I_w^\star)^{-1}$ (Corollary 7.7). We fix a maximal split torus $T \subset G$ and a nice open compact subgroup $K \subset G$ (in the sense of [2]) in good relative position to T. Recall that this means that

$$K = K^+ \cdot K^0 \cdot K^- \qquad (10.10.1)$$

for any pair of opposite Borel subgroups $B^+ = N^+T$, $B^- = N^-T$, where $K^+ = N^+ \cap K$, $K^- = N^- \cap K$, $K^0 = T \cap K$. We fix a Borel subgroup containing T, this defines the cone of dominant coweights $X_*(T)^+ \subset X_*(T)$.

The following technical statement is not hard but plays a role in Bernstein's theory.

The choice of a uniformizer π yields a homomorphism from the coweight lattice $\iota : X_*(T) \to T$. For $\nu \in X_*(T)^+$ let $e_\nu \in \mathcal{H}$ be the delta-function of the two-sided coset $K\iota(\nu)K$.

Recall the map $\iota_X = \iota_\Sigma : X_*(T) \to X$. Let $x_\nu \in C(X)$ be the delta-function of the $K \times K$ orbit of $\iota_X(\nu)$. We set $\theta_\nu = B(x_\nu)$.

Proposition 10.1. *The map $X_*(T) \to \mathcal{H}$, $\nu \mapsto \theta_\nu$ is uniquely characterized by the following two properties.*
i) $\theta_{\mu_1+\mu_2} = \theta_{\mu_1}\theta_{\mu_2}$.
ii) There exists ν_0 such that for $\nu \in \nu_0 + X_*(T)^+$, we have $\theta_\nu = e_\nu$.

We will need the following Lemma.

Lemma 10.2. [2] *a) For $\mu, \nu \in X_*(T)^+$ we have $e_\mu e_\nu = e_{\mu+\nu}$.*
b) Given $\lambda \in X_*(T)^+$, there exists $N > 0$ such that the kernel of left multiplication by $e_{N\lambda}$ on $\mathcal{H}_K$ equals the kernel of left multiplication by $e_{(N+1)\lambda}$.

Proof. (a) follows directly from (10.10.1). (b) follows from the fact that $\mathcal{H}_K$ is right Noetherian. $\qquad \square$

Proof of Proposition. To check uniqueness, assume θ_μ and θ'_μ are different collections of elements satisfying (i,ii). Fix $\nu \in X_*(T)^+$ such that $\theta_\nu = e_\nu = \theta'_\nu$. We have also $\theta_{\mu+N\nu} = e_{\mu+N\nu} = \theta'_{\mu+N\nu}$ for some N. Thus $\theta_\mu - \theta'_\mu$ lies in the image of left multiplication by e_ν, as well as in the kernel of left multiplication by $e_{N\nu}$. Lemma 10.2(b) implies that $\theta_\mu = \theta'_\mu$, which proves uniqueness.

It remains to check that $\theta_\nu = B(x_\nu)$ does satisfy (i,ii). It has been verified above that given an admissible bijection between $U \subset G$ and $V \subset X$ there exists $\nu_0 \in X_*(T)^+$ such that the bijection sends the set $K\iota_G(\nu)K$ into the set $K\iota_X(\nu)K$ for $\nu \in \nu_0 + X_*(T)^+$. Thus (ii) follows from the construction of B. Given $\nu \in X_*(T)^+$, Proposition 3.9 implies that there exists $\nu_0 \in X_*(T)^+$ such that

$$e_\nu * x_\mu = x_{\nu+\mu} = x_\mu * e_\nu \qquad (10.10.2)$$

for $\mu \in \nu_0 + X_*(T)^+$. Since the elements x_μ are permuted by the action of $\mathbb{T}$ commuting with the action of $\mathcal{H} \otimes \mathcal{H}$, we see that (10.10.2) holds for

all μ. Hence $e_\nu \theta_\mu = \theta_{\nu+\mu}$ for all μ. We now proceed to check (i). Pick ν such that $e_\eta = \theta_\eta$ when η is either $\nu + \mu_1$, or $\nu + \mu_1 + \mu_2$. Then we see that $e_\nu \theta_{\mu_1+\mu_2} = e_{\nu+\mu_1+\mu_2} = e_\nu \theta_{\mu_1} \theta_{\mu_2}$. Also, both $\theta_{\mu_1+\mu_2}$ and θ_{μ_1}, θ_{μ_2} lie in the image of left multiplication by $e_{N\nu}$ for all N. Thus the desired equality follows from Lemma 10.2(b). $\square$

Remark 10.1. The case most often encountered in the literature is when $K = I$ is the Iwahori subgroup. Then the elements θ_ν are invertible elements in the affine Hecke algebra. They form a part of a system of generators for this algebra discovered by Bernstein [10], which have found numerous important applications.

Now Corollary 7.7 shows that

$$\theta_\nu(g) = B(x_\nu)(g) = A^\star I_1(g). \qquad (10.10.3)$$

This yields a spectral expression for the Plancherel functional $f \mapsto f(1)$ restricted to the subalgebra $A_K \subset \mathcal{H}$ spanned by θ_ν. To state it introduce the following notation.

For a character χ of T let $i(\chi) = i_B^G(\chi)$ denote the (unitary) induction. We define a rank one operator $\Pi : i(\chi)^K \to i^-(\chi)^K$ as follows. Notice that the space $i(\chi)^K$ splits as a direct sum of one dimensional subspaces indexed by the set $K\backslash G/B$; likewise, $i^-(\chi)$ is a direct sum of one dimensional subspaces indexed by $K\backslash G/B^-$. The one dimensional summands corresponding to the coset of 1 in the two spaces are canonically isomorphic (both are identified with the space of one dimensional representation χ of T), let us denote this space by $\mathbb{C}_\chi$. We define $\Pi : i(\chi)^K \to i^-(\chi)^K$ as the composition $i(\chi)^K \to \mathbb{C}_\chi \to i^-(\chi)^K$, where the first arrow is the projection arising from the above splitting into a direct sum, and the second arrow is the embedding.

Let $Char$ denote the space of characters of the torus $\mathbb{T}$ viewed as an affine ind-algebraic variety over $\mathbb{C}$. We have $Char \cong \bigsqcup_{\chi \in \mathbb{T}(O)\check{}} Char_\chi$, where $\mathbb{T}(O)\check{}$ is the discrete set of characters of $\mathbb{T}(O)$ and $Char_\chi$ is the space of characters whose restriction to $T(O)$ is equal to χ. Then $Char_\chi$ is a principal homogeneous space for the dual torus $\check{\mathbb{T}}$ over $\mathbb{C}$. Let $\mathcal{O}^f(Char) = \bigoplus_\chi \mathcal{O}(Char_\chi)$ denote the (non-unital) ring of regular functions on $Char$ which vanish on all but a finite number of components. We have a canonical isomorphism $\mathcal{O}^f(Char) \cong \mathbb{C}(\mathbb{T})$, and a noncanonical isomorphism $\mathcal{O}^f(Char) \cong \bigoplus_{\chi \in \mathbb{T}(O)\check{}} \mathcal{O}(\check{\mathbb{T}})$.

Recall the partial compactification $\overline{\check{\mathbb{T}}}$ of the torus $\check{\mathbb{T}}$ given by $\overline{\check{\mathbb{T}}} = Spec(\mathbb{C}[X^*(\check{\mathbb{T}})^+])$ where $X^*(\check{\mathbb{T}})^+ \subset X^*(\check{\mathbb{T}})$ is the semigroup of dominant weights. Let $C_b(\mathbb{T})$ be the space of distributions invariant with respect to an open subgroup in $\mathbb{T}$ and whose support is contained in a compact subset of $\overline{\check{\mathbb{T}}}$. To give a spectral description of this ring introduce the affine toric variety $\overline{\check{\mathbb{T}}} = Spec(X_*(\check{\mathbb{T}})^+)$, where $X_*(\check{\mathbb{T}})^+$ is the dual cone to $X^*(\check{\mathbb{T}})^+$, i.e. $X^*(\check{\mathbb{T}})^+ \subset X_*(\check{\mathbb{T}}) = X^*(\mathbb{T})$ is the set of weights which are positive

rational linear combinations of simple roots of $\mathbb{T}$. This defines a partial compactification $\overline{Char}_\chi = (\overline{\mathbb{T}} \times Char_\chi)/\mathbb{T}$ of the principal homogeneous space $Char_\chi$. Let $\widehat{\mathcal{O}}(Char_\chi)$ be the ring of functions on the punctured formal neighborhood of the divisor $\partial Char_\chi = \overline{Char}_\chi \setminus Char_\chi$. Set $\widehat{\mathcal{O}}(Char)^f = \bigoplus_\chi \widehat{\mathcal{O}}(Char_\chi)$. Then it is easy to see that $\widehat{\mathcal{O}}(Char)^f \cong C_b(\mathbb{T})$.

Choosing a point in $Char_\chi$ we get an identification $Char_\chi \cong \mathbb{T}$, $\overline{Char}_\chi \cong \overline{\mathbb{T}}$, $\widehat{\mathcal{O}}_\chi \cong \{\phi : X_*(\mathbb{T}) \to \mathbb{C} \mid \mathrm{supp}(\phi) \subset S + X_*(\mathbb{T})^+ \text{ for a finite set } S \subset X_*(\mathbb{T})\}$. We define a linear functional $\int : \widehat{\mathcal{O}}_\chi \to \mathbb{C}$ sending a function $\phi : X_*(\mathbb{T}) \to \mathbb{C}$ to $\phi(0)$. It is easy to see that this is independent of the choice of a point in $Char_\chi$. If $f \in \widehat{\mathcal{O}}(Char_\chi)$ is a Laurent expansion of a rational function on $Char_\chi$ (we assume that the divisor of poles of f does not pass through the point orbit $\{t_0\}$ of $\mathbb{T}$ on $\overline{Char}_\chi$, so that the Laurent expansion is well defined), then $\int f$ is the integral of f over a coset of the maximal compact subtorus close to t_0, hence the notation. We let $\int : \widehat{\mathcal{O}}(Char)^f \to \mathbb{C}$ be the linear functional coinciding with the above functional on $Char_\chi$ for every χ. It is easy to see that under the isomorphism $\widehat{\mathcal{O}}(Char)^f \cong C_b(\mathbb{T})$, the functional $\int$ goes to the functional $h \mapsto h(1)$.

In the next Theorem we consider a function $\chi \mapsto Tr(h\Pi R_{w_0}^{-1}, i_\chi)$, $h \in \mathcal{H}$. This is a rational function on $Char$ vanishing on all but a finite number of components. Moreover, in view of Proposition 8.1, for every χ the point $t_0 \in \overline{Char}_\chi$ does not lie in the divisor of poles of this function. Thus the functional $\int$ is well defined on such a function.

Theorem 10.3. *For $h \in A_K$ we have*

$$h(1) = \int Tr(h\Pi R_{w_0}^{-1}, i_\chi).$$

Proof. We can assume that $h = \theta_\nu$. Applying Corollary 7.7 with $w = 1$, we get $\theta_\nu = A^* I^{-1}(x_\nu)$.

A function $\phi \in C_b^{w=1}(Y)$ defines an operator on a completion of the universal principal series, $C(G/U) \otimes_{\mathcal{O}(Char)} \widehat{\mathcal{O}}(Char)$.

By standard Fourier analysis, we have

$$\int Tr(i_\chi(\phi))d\chi = \int_{pr_Y^{-1}(\Delta_\mathcal{B})} \phi,$$

where $\Delta_\mathcal{B} \subset \mathcal{B}^2$ is the diagonal, and integral in the left hand side has the same meaning as in the statement of the Theorem. It is clear that the integral in the right hand side equals $A^*(\phi)|_1$.

Thus we will be done if we show that

$$i_\chi(I_{w=1}(x_\nu)) = i_\chi(\theta_\nu)\Pi R_{w_0}^{-1}. \tag{10.10.4}$$

A function $f \in \mathbb{C}(X)$ defines an operator between the principal series and the opposite principal series, $\sigma_\chi(f) : i_\chi \to i_\chi^-$. In particular, it is easy

to see that $\sigma_\chi(x_0) = \Pi$ for all χ. Also, it follows from the definitions that $i_\chi(I_{w=1}^{-1}(f)) = R_{w_0}^{-1} \circ \sigma_\chi(f)^{-1}$. Finally, using that all the maps involved commute with the $\mathcal{H} \otimes \mathcal{H}$ action, we get:

$$i_\chi(I_{w=1}(x_\nu)) = i_\chi(\theta_\nu * x_0) = i_\chi(\theta_\nu)\Pi R_{w_0}^{-1}.$$

Thus we have proven (10.10.4), and hence the Theorem.   □

Remark 10.2. In the case when $K = I$ is the Iwahori subgroup a formula similar to Theorem 10.3 appears in [11], [12].

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11. Appendix: quasi-normal cone for toric coverings

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In this section we introduce a version of the definition of a normal cone which behaves well for a class of singular varieties, including De Concini - Procesi compactifications of not necessarily adjoint groups. See [MT] for another approach to wonderful compactification of non-adjoint groups.

11.1. **Quasi-normal cone.** Let X be a smooth variety over a field F and $Z \subset X$ be a smooth locally closed subvariety. Let X' be a normal scheme and $X' \to X$ be a finite flat morphism, let Z' be the preimage of Z equipped with reduced subscheme structure. Let $N_X(Z)$ be the normal bundle.

Let $\widetilde{N}_X(Z)$ be the deformation to the normal cone; thus $(\mathbb{A}^1 \setminus \{0\}) \times X \subset \tilde{N}_X(Z) \supset N_X(Z)$. Recall that $\widetilde{N}$ comes equipped with a $\mathbb{G}_m$ action which dilates the fibers of the normal bundle and acts on $(\mathbb{A}^1 \setminus \{0\}) \times X$ by $t : (z, x) \mapsto (tz, x)$.

Let $\widetilde{N}_{X'}(Z')$ be the normalization of $\widetilde{N}_X(Z)$ in $(\mathbb{A}^1 \setminus \{0\}) \times X'$, and set $N_{X'}(Z') = \widetilde{N}_{X'}(Z) \times_{\mathbb{A}^1} \{0\}$. We call $N_{X'}Z'$ the *quasi-normal cone* of Z' in X'.

We have a locally closed embedding $Z' \times \mathbb{A}^1 \to \widetilde{N}_{X'}Z'$.

Definition. We say that X' is *well approximated around Z* if the following conditions hold.

(a) For some $d > 0$ the composition of the natural action of $\mathbb{G}_m$ on $\widetilde{N}_Z(X)$ with the map $\mathbb{G}_m \to \mathbb{G}_m$, $t \mapsto t^d$, lifts to an action α_d of $\mathbb{G}_m$ on $\widetilde{N}_{Z'}(X')$.

(b) Zariski locally on Z there exists an isomorphism τ between the formal neighborhoods of $Z' \times \mathbb{A}^1$ in $N_{X'}(Z') \times \mathbb{A}^1$ and in $\widetilde{N}_{Z'}(X')$, such that

   (i) τ restricted to the preimage of $0 \in \mathbb{A}^1$ equals identity.

   (ii) τ is $\mathbb{G}_m$-equivariant where $\mathbb{G}_m$ acts on $\widetilde{N}_{Z'}(X')$ via α_d and on $N_{X'}(Z') \times \mathbb{A}^1$ by $t : (x, z) \mapsto (\alpha_d(x), t^d z)$.

If F is a local field, we can repeat the definition replacing an isomorphism of formal neighborhoods by an analytic isomorphism of actual $F^\times$ invariant neighborhoods in the space of F-points; "Zariski local" should then be replaced by local in the sense of F-topology. We will use the term "analytically well approximated" for this version of the property. If F is non-Archimedian, existence of local isomorphisms implies existence of a global isomorphism of appropriate neighborhoods.

In the latter case, the restriction of τ to the fiber over $1 \in \mathbb{A}^1$ will be called an *admissible bijection*.

Example 11.1. If $X' = X$, then giving τ amounts to giving an isomorphism between a neighborhood of Z in X and a neighborhood of the zero section in the normal bundle $N_X(Z)$, whose differential in the normal direction equals identity.

11.2. **Toric covering.** Let X be again a smooth variety over F, and let $D_i, i \in I$ be smooth divisors with normal crossing in X. For a subset $J \subset I$ we have the corresponding stratum $X_J = \cap D_i, i \in J$. We fix $J \subset I$ and let $Z = X_J$.

We have line bundles $\mathcal{L}_i = \mathcal{O}(D_i)$ on X, each coming with a canonical section s_i. They can be combined into a T-bundle $\mathcal{E}$ for the torus $T = (\mathbb{G}_m)^I$.

Let T' be another torus and $T' \to T$ be a fixed isogeny. Suppose that the above T-bundle on X lifts to a T'-bundle $\mathcal{E}'$ (i.e. $\mathcal{E}$ is the push-forward of $\mathcal{E}'$); we fix such a T'-bundle $\mathcal{E}'$. This data defines a ramified covering $X' \to X$ as follows.

Let $A = (\mathbb{A}^1)^I$, thus T acts on A making it a toric variety. The T-bundle $\mathcal{E}$ defines an associated bundle $A_{\mathcal{E}}$ with the fiber A; of course $A_{\mathcal{E}}$ is nothing but the total space of the sum of line bundles $\mathcal{L}_i$; the sections s_i combine to a section $s : X \to A_{\mathcal{E}}$.

Let A' be the normalization of A in the covering $T' \to T$. We also can form an associated bundle $A'_{\mathcal{E}'}$. Set $X' = X \times_{A_{\mathcal{E}}} A'_{\mathcal{E}'}$. We call X' obtained by this construction a toric covering.

Example 11.2. Let G be a reductive algebraic group. Let $G_{ad} = G/Z(G)$, where $Z(G) \subset G$ is the center, and let $G' \subset G$ be the derived (commutator) subgroup. Let $X = (G/G') \times \overline{G}_{ad}$ where $\overline{G}_{ad}$ is the wonderful compactification of G_{ad}, and D_i be the components of the complement $X \setminus (G/G') \times G_{ad}$. The torus T is identified with the (abstract) Cartan subgroup of G_{ad}. The total space of $\mathcal{E}$ is easily seen to be identified with $(G/G') \times S^0(G_{ad})$, where $S^0(G_{ad}) \subset S(G_{ad})$ is the smooth locus in Vinberg's semi-group $S(G_{ad})$. Let T' be the abstract Cartan of G', and let $\mathcal{E}$ be the T-bundle whose total space is $S^0(G)$. [If G is simply-connected, this is a universal torus bundle over X, i.e. the map from $X^*(T)$ to $Pic(X)$ given by this bundle is an isomorphism; we neither prove nor use this fact]. It is easy to see that the preimage of $(G/G') \times G_{ad} \subset X$ in X' is isomorphic to G. It follows from the Theorem below that X' is the normalization of $(G/G') \times \overline{G}_{ad}$ in G.

Example 11.3. Assume that the line bundles $\mathcal{L}_i$ are trivial (i.e. D_i is cut out by a global functions). Fix $\mathcal{E}'$ to be the trivial T'-torsor. Then s amounts to a smooth morphism $X \to A$, so that Z is the preimage of the closure of T-orbit in A; we have $X' = X \times_A A'$.

11.3. **Theorem.** *Let $X' \to X \supset D_i$ be a toric covering and $Z = X_J$ as above.*

a) X' is a normal variety and the map $X' \to X$ is finite and flat.

b) X' is well approximated around Z. If F is a local field it is also analytically well approximated.

c) The quasi-normal cone $N_{X'}(Z')$ is canonically isomorphic to $A'_{\mathcal{E}'} \times_X Z$.

Remark. If $T' = T$, so that $X' = X$, the isomorphism of part (c) amounts to "adjunction formula" $\mathcal{O}(D)|_D \cong N_X(D)$.

Proof. It is easy to check that all the definitions above are compatible with smooth base change. Thus it suffices to check that the statements hold after base change with respect to the morphism from the total space of the torsor $\mathcal{E}$ to X. This reduces the proof to the situation of Example 11.3. Using compatibility with smooth base change again, we see that it suffices to show that A' is normal, $A' \to A$ is flat and finite; that A' is well approximated around the closure A_J of a T-orbit in A, and that $N_{A'}(A'_J) = A'$. Normality

and finiteness are clear from the definition, and flatness is easy to show. The deformation to the normal cone $\widetilde{N}_A(A_J)$ is constant, which implies the rest. □

Remark. Part (c) of the Theorem shows that an open part of $N_{X'}(Z')$ is identified with the total space of $\mathcal{E}'|_Z$. In the situation of Example 11.2, this is the space denoted by X_J in the paper.

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